

Hyper-Kähler Geometry and Invariants of Three-Manifolds

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Abstract. We study a 3-dimensional topological sigma-model, whose target space is a hyper-Kähler manifold X . A Feynman diagram calculation of its partition function demonstrates that it is a finite type invariant of 3-manifolds which is similar in structure to those appearing in the perturbative calculation of the Chern-Simons partition function.

The sigma-model suggests a new system of weights for finite type invariants of 3-manifolds, described by trivalent graphs. The Riemann curvature of X plays the role of Lie algebra structure constants in Chern-Simons theory, and the Bianchi identity plays the role of the Jacobi identity in guaranteeing the so-called IHX relation among the weights.

We argue that, for special choices of X , the partition function of the sigma-model yields the Casson-Walker invariant and its generalizations. We also derive Walker's surgery formula from the $SL(2, \mathbf{Z})$ action on the finite-dimensional Hilbert space obtained by quantizing the sigma-model on a two-dimensional torus.

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1 Introduction

The invariants of 3-dimensional topology which are based on quantum Chern-Simons theory [1] contain a wealth of information, in some ways almost too much. One would like to be able to extract simple statements and in particular to compare these invariants to more classical invariants of topology. For this one might want something more elementary than the full-fledged Chern-Simons theory.

A possible approach to study the structure of the quantum invariants is to look at their semi-classical asymptotics. Let us recall that from the quantum field theory point of view, one constructs quantum invariants of three-manifolds with the following starting point. One introduces a gauge group G , usually compact, and one considers a connection A on a G -bundle E over an oriented three-manifold M , with the Chern-Simons action

$$S_{\text{CS}} = \frac{1}{2} \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (1.1)$$

Here Tr is a quadratic form on the G Lie algebra with certain integrality properties. The quantum invariant of M (also known as the partition function) is expressed as a path integral over (the gauge equivalence classes of) connections A

$$Z_{\text{CS}}(M; k) = \int \exp \left(\frac{ik}{2\pi} S_{\text{CS}} \right) \mathcal{D}A; \quad (1.2)$$

here k is an integer. As the definition of S_{CS} requires no metric on M , the partition function (1.2) is formally expected to be a topological invariant of M . Closer investigation shows (at a physical level of rigor) that this is actually so, modulo some subtleties about a framing of M that must be introduced in the quantization.

The form of the integrand in eq. (1.2) suggests that the behavior of $Z_{\text{CS}}(M; k)$ in the semi-classical regime (i.e. near $k = \infty$) should be governed by a stationary phase expansion. The starting point of such an expansion is to pick a critical point A_0 of S_{CS} , that is to say, a flat connection on E . Then one expands the quantum theory about $A = A_0$, generating Feynman diagrams in a standard fashion [2], [3], [4], [5]. (The analysis is most simple if the flat connection A_0 is isolated modulo gauge transformations; otherwise one meets some additional complications, analyzed in this context in [6].) The asymptotic expansion around the c^{th} flat connection (or more generally, the c^{th} component of the moduli space of flat connections) produces an asymptotic series $Z_{\text{CS}}^{(c)}(M; k)$ in powers of $1/k$. The exact Chern-Simons invariant $Z_{\text{CS}}(M; k)$ (which is defined as an actual function of k , at least for positive integer arguments, in contrast to the individual $Z_{\text{CS}}^{(c)}(M; k)$ which may very possibly only exist in general as asymptotic series) has to all orders in $1/k$ an asymptotic expansion of the form

$$Z_{\text{CS}}(M; k) = \sum_c Z_{\text{CS}}^{(c)}(M; k). \quad (1.3)$$

Each of the $Z_{\text{CS}}^{(c)}(M; k)$ is the product of the “classical exponential” $\exp\left(\frac{ik}{2\pi} S_{\text{CS}}^{(c)}\right)$ (here $S_{\text{CS}}^{(c)}$ is a Chern-Simons invariant of a classical flat connection in the c^{th} component) multiplied by some determinants and by an asymptotic series in k^{-1} . In this expansion, the term of relative order k^{-n} in the asymptotic series is expressed in terms of trivalent graphs with $n+1$ loops. The graphs are trivalent because S_{CS} is a cubic function of A .

If the flat connection A_0 is non-trivial, one really constructs in this way an invariant of the three-manifold M together with a representation of its fundamental group in G (which determines the flat connection). If one wants to get a “pure” three-manifold invariant, one approach is to take A_0 to be the trivial flat connection and consider its contribution $Z_{\text{CS}}^{(\text{tr})}(M; k)$ in the sum (1.3). One finds that the G -dependence of the three-manifold invariant $Z_{\text{CS}}^{(\text{tr})}(M; k)$ can be described very simply, provided that M is a rational homology sphere, which ensures that the trivial flat connection is isolated. (Otherwise, the trivial flat connection is a point on a moduli space of flat connections over which one must integrate, and things become more complicated.) In fact, many of the ingredients are independent of

G . To each $n + 1$ -loop trivalent graph Γ , one associates a certain fairly complicated integral I_Γ of a product of Green's functions over a product $M \times M \times \dots \times M$. $I_\Gamma(M)$ depends on a metric on M (or some other “gauge fixing” data that break the topological symmetry), but not on G . One also associates to each such graph a “weight factor” $a_\Gamma(G)$ which depends on G but not on M . The trivial connection contribution is proportional to the (exponential) of an asymptotic series

$$\exp \left(\sum_{n=1}^{\infty} S_{n+1,G}(M) k^{-n} \right) \quad (1.4)$$

The coefficient $S_{n+1,G}(M)$ in the asymptotic expansion of $Z_{\text{CS}}^{(\text{tr})}(M; k)$ for gauge group G is then

$$S_{n+1,G}(M) = \sum_{\Gamma \in \Gamma_{n,3}} a_\Gamma(G) I_\Gamma(M), \quad (1.5)$$

where the sum runs over all trivalent graphs Γ with $n + 1$ loops (and $2n$ vertices). The Jacobi identity of the Lie algebra of G is used to show that although the individual integrals $I_\Gamma(M)$ depend on the metric, the metric-dependence cancels out of the sum.

Part of the interest in the coefficients $S_{n,G}(M)$ is related to the fact that they are expected to be related in a relatively accessible fashion to classical invariants, but with the information organized in a new way suggested by quantum field theory. Indeed, the coefficient in front of the exponential (1.4) is known to be related to the order of the first homology group $|H_1(M, \mathbf{Z})|$ [7] and the second coefficient is proportional to the Casson invariant of M [8], [9]¹.

The asymptotic coefficients $S_{n,G}(M)$ of $Z_{\text{CS}}^{(\text{tr})}(M; k)$ fall into the category of the so-called “finite type” invariants of 3-manifolds M , which were introduced in [10] (see [11] for an exhaustive review of the properties of finite type invariants of knots). Loosely speaking, these invariants can detect the features of M which are only of limited (i.e. finite) complexity. The coefficient $S_{n,G}(M)$ is finite type of order $3(n - 1)$.

Let $\Gamma_{n,3}$ be the set of all closed graphs with $2n$ trivalent vertices. We assume that there is a cyclic ordering of legs at each vertex. A function on $\Gamma_{n,3}$ is called a weight if it is

¹In the case of Seifert rational homology spheres, the relation between the Casson invariant and two-loop perturbative correction was observed independently by J. Andersen by analyzing some results of [8].

antisymmetric under the permutation of legs at a vertex and if it satisfies the so-called IHX relation (see e.g. [11] for its definition). It was shown in [12] that for every weight function one can construct a finite type invariant of order $3n$. The weight functions of the coefficients of $Z_{\text{CS}}^{(\text{tr})}(M; k)$ are calculated by placing the structure constants of the Lie group of G at the vertices of trivalent graphs and contracting their indices along the edges with the help of quadratic form Tr . The IHX relation for these weights follows from the Jacobi identity.

There has been considerable interest in the question of whether there are other weight factors b_Γ , not derived from Lie groups, such that $\sum_\Gamma b_\Gamma I_\Gamma$ is likewise an invariant of rational homology spheres. There has also been much interest in relating these graphical invariants to three-manifold invariants constructed in other ways.

In the present paper, we will obtain two results bearing on these questions:

- (I) For any compact or asymptotically flat hyper-Kähler manifold X of real dimension $4n$, we construct weight functions $b_\Gamma(X)$ (see eq. (3.41)) on closed trivalent graphs with up to $2n$ vertices. We also construct weight functions on trivalent graphs which have external legs. We provide a simple mathematically rigorous proof that the weights satisfy the IHX relation, thus demonstrating that the linear combinations of integrals $\sum_{\Gamma \in \Gamma_{n,3}} b_\Gamma(X) I_\Gamma(M)$ are topological invariants of rational homology spheres M . As in the Chern-Simons case, there is also a corresponding statement for general oriented 3-manifolds.²
- (II) We show that the (unique) invariant that can be constructed from the integrals $I_\theta(M)$ associated with the two-loop graph (we call this graph θ) is proportional to the $SU(2)$ Casson-Walker invariant. Although we do not obtain precise formulas, our construction shows that the analog of Casson-Walker invariant based on a Lie group of rank $r > 1$ [13] is a linear combination of integrals $I_\Gamma(M)$ related to trivalent graphs with $r + 1$ loops.

²There is a difference, though: in the Chern-Simons case increasing b_1 makes the analysis more complicated, while here it becomes more simple.

Each of these two results can be understood, to a certain extent, without the machinery we will develop. In particular, once one suspects that it is true, (I) can be checked by a direct calculation. Similarly, the result (II) is not new, having been obtained before [8], [9] by analyzing the surgery formula for the partition function of $SU(2)$ Chern-Simons theory. However we hope to show that these results have a very natural origin in a certain quantum field theory context.

The natural context is in fact a new type of 3-dimensional topological field theory. This theory is a twisted version of an $N = 4$ supersymmetric sigma-model. The sigma-model partition function $Z_X(M)$ is expressed as a path integral over the maps from M into a (compact or asymptotically flat) hyper-Kähler manifold X . For a $4n$ -dimensional hyper-Kähler manifold X , a perturbative calculation of $Z_X(M)$ presents it as a sum over closed graphs with $2n(3 - b_1(M))$ -valent vertices

$$Z_X(M) = \sum_{\Gamma \in \Gamma_{n, 3-b_1(M)}} b_\Gamma(X) I_\Gamma(M). \quad (1.6)$$

Here $b_1(M) = \dim H_1(M, \mathbf{R})$ is the first Betti number of M . For M with $b_1(M) = 0$, $b_\Gamma(X)$ present new weight functions on trivalent graphs. Instead of Lie algebras, they are based on compact or asymptotically flat hyper-Kähler manifolds. Roughly speaking, the Riemann curvature of X plays the role of Lie algebra structure constants in writing the expression for $b_\Gamma(X)$, and the Bianchi identity plays the role of the Jacobi identity in the proof of the IHX relation. This explains our first claim.

The claim (II) has a perhaps more subtle origin than (I). $N = 2$ supersymmetric gauge theory in four dimensions with gauge group $SU(2)$ can be topologically twisted [14] to get a theory whose correlation functions are Donaldson invariants of four-manifolds. If dimensionally reduced to three dimensions, this theory has a twisted version with the Casson-Walker invariant for its partition function (see, e.g. [15] for a review). On the other hand, this three-dimensional theory reduces at low energies [16] to a sigma-model whose target space is a certain smooth, non-compact but asymptotically flat four-dimensional hyper-Kähler manifold X_{AH} , which in fact coincides with the two-monopole moduli space whose geometry was described in detail in [17]. So the Casson-Walker invariant coincides with the

invariant computed from the sigma-model with target space X_{AH} ; this invariant is a multiple of the unique two-loop graphical invariant, explaining our second claim.

If $SU(2)$ is replaced by a gauge group G of rank r , the two-monopole moduli space would simply be replaced by the moduli space of vacua of the three-dimensional supersymmetric gauge theory (with $N = 4$ supersymmetry in three dimensions, corresponding to $N = 2$ in four dimensions) with gauge group G . This moduli space is of dimension $4r$, so the Casson-Walker invariant of G for rational homology spheres should be a finite type invariant of order $3r$ which is computable from trivalent graphs with $r + 1$ loops (but we do not know precisely which weights are required). In fact, it has recently been argued that the moduli space for $G = SU(n)$ is simply the reduced moduli space of BPS n -monopoles of $SU(2)$ [18], [19]. This fact could be used, in principle, to obtain detailed formulas for $SU(n)$.

In Section 4, we will compare in detail the invariant we obtain for the case where the target is X_{AH} to the Casson-Walker invariant, as extended to arbitrary three-manifolds by Lescop [20]. The details depend very much on the value of the first Betti number $b_1(M)$. We find that the invariant $Z_{X_{\text{AH}}}(M)$ computed from the sigma-model and the Casson-Walker-Lescop invariant agree up to a sign factor

$$Z_{X_{\text{AH}}}(M) = \frac{1}{2}(-1)^{b_1} \lambda_{\text{C}}(M). \quad (1.7)$$

Here $\lambda_{\text{C}}(M)$ is *twice* the Casson invariant as defined in [20] (roughly speaking, in our normalization $\lambda_{\text{C}}(M)$ measures the Euler characteristic of the moduli space of flat $SU(2)$ connections on M).

For compact X , we expect that the theory under consideration in this paper will obey the full axioms of a (non-unitary) topological quantum field theory as formalized by Atiyah [21]. (Unitarity would mean that the vector spaces \mathcal{H}_{Σ} associated to two-manifolds Σ have hermitian metrics compatible in a natural way with the rest of the data; that is so in Chern-Simons theory but not in the theory considered here.) Proving this goes beyond the scope of the present paper, though we develop many of the relevant facts in Section 5. We suspect that a full direct proof would be far simpler than the corresponding analysis of Chern-

Simons theory [22]. If X is not compact, one cannot quite get this structure, as we will see in Section 5.

In the analysis, we will meet two “anomalies” which are “curable.” The path integral representation of the partition function $Z_X(M)$ requires that M be equipped with a 2-framing and an orientation on the sum of (co)homology spaces

$$2H^0(M, \mathbf{R}) \oplus 2H^1(M, \mathbf{R}); \quad (1.8)$$

here $2H$ denotes $H \oplus H$. However, since there is a canonical choice of 2-framing [23] and a canonical choice of orientation for the space (1.8), the partition function $Z_X(M)$ can be transformed into a genuine invariant of an oriented 3-manifold.

The problem treated in the present paper is the three-dimensional analog of “integrating over the u -plane” in Donaldson theory. Some of the features we find should have analogs for four-manifolds with small values of b_1 and b_2^+ .

2 The topological sigma-model

2.1 Review of hyper-Kähler geometry

We begin with a review of those few aspects of hyper-Kähler geometry which we will need.

A hyper-Kähler manifold X is a manifold of real dimension $4n$

$$\dim_{\mathbf{R}} X = 4n \quad (2.1)$$

($4n$ will denote the dimension of X throughout the paper). X has a Riemannian metric such that the holonomy of the Levi-Civita connection lies in an $Sp(n)$ subgroup of $SO(4n)$. This means that the complexification of the tangent bundle TX of X decomposes as $TX \otimes_{\mathbf{R}} \mathbf{C} = V \otimes S$, where V is a rank $2n$ complex vector bundle with structure group $Sp(n)$, and S is a trivial rank two bundle. In fact, $SO(4n)$ contains a subgroup $(Sp(n) \times SU(2))/\mathbf{Z}_2$; $Sp(n)$ acts on V and $SU(2)$ (which equals $Sp(1)$) acts on S . The Levi-Civita connection on $TX_{\mathbf{C}}$

simply reduces to a $Sp(n)$ connection on V (times the trivial connection on S). The bundle V will appear throughout the paper.

Sometimes, we will pick on X local coordinates ϕ^i ; the metric of X will be called g_{ij} . The indices $A, B, \dots = 1, 2$ will label the two-dimensional representation of $Sp(1)$, while the indices $I, J, \dots = 1, \dots, 2n$ will refer to the $2n$ -dimensional representation of $Sp(n)$.

The trivial $SU(2)$ bundle S is endowed with an $SU(2)$ -invariant antisymmetric tensor ϵ_{AB} . The antisymmetric tensor ϵ^{AB} is defined as the inverse of ϵ_{AB}

$$\epsilon^{AB}\epsilon_{BC} = \delta_C^A. \quad (2.2)$$

Similarly, the fact that V has structure group $Sp(n)$ means that there is an invariant antisymmetric tensor ϵ_{IJ} in $\wedge^2 V$. We write its inverse as ϵ^{IJ} , defined so that

$$\epsilon^{IJ}\epsilon_{JK} = \delta_K^I. \quad (2.3)$$

The ϵ tensors are used to raise and lower $Sp(1)$ and $Sp(n)$ indices (it will hopefully cause no confusion to refer to both of these invariant tensors as ϵ). Finally, the decomposition $TX_C = V \otimes S$ corresponds to the existence of covariantly constant tensors γ_i^{AI} and γ_{AI}^i that describe the maps from $V \otimes S$ to TX and vice-versa.

Now let us describe the form of the Riemann tensor on a hyper-Kähler manifold. In general, the Riemann tensor is a two-form R_{ij} ($= -R_{ji}$) valued in the Lie algebra of $SO(4n)$. The condition of $Sp(n)$ holonomy means that R actually takes values in the Lie algebra of $Sp(n)$. This is equivalent to the statement that $R_{ij} = \gamma_i^{AI}\gamma_j^{BJ}\epsilon_{AB}R_{IJ}$, where $R_{IJ} = R_{JI}$ is a two-form with values in $\text{Sym}^2 V$. (Recall that the Lie algebra of $Sp(n)$ can be identified with the *symmetric* matrices $M_{IJ} = M_{JI}$.) Now let us write explicitly $R_{ij} = \frac{1}{2}R_{ijkl}dx^k \wedge dx^l$. The symmetry of the Riemann tensor with $R_{klij} = R_{ijkl}$ means that we can make the same manipulation on the first two indices that we just made on the last two, learning that

$$R_{ijkl} = -\gamma_i^{IA}\gamma_j^{JB}\gamma_k^{KC}\gamma_l^{LD}\epsilon_{AB}\epsilon_{CD}\Omega_{IJKL}, \quad (2.4)$$

for some $\Omega \in \text{Sym}^2 V \otimes \text{Sym}^2 V$ (that is, $\Omega_{IJKL} = \Omega_{JIKL} = \Omega_{IJLK}$). Finally, the property of the Riemann tensor that $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ implies that Ω is completely symmetric in all indices.

2.2 Bianchi identity

Now, armed with this knowledge of the structure of the Riemann tensor, let us look at the Bianchi identity, which in general reads $DR_{ij} = 0$ (D being the exterior derivative defined using the Levi-Civita connection). It reduces in the hyper-Kähler case to

$$D_{AI}\Omega_{JKLM} = D_{AJ}\Omega_{IKLM}, \quad (2.5)$$

where $D_{AI} = \gamma_{AI}^i D_i$.

Now, consider the second derivative of Ω , antisymmetrized on the $SU(2)$ and $Sp(n)$ indices:

$$\epsilon^{AB} (D_{AI}D_{BJ} - D_{AJ}D_{BI}) \Omega_{KLMN}. \quad (2.6)$$

To analyze this expression, we use first (2.5) to write $D_{AI}D_{BJ}\Omega_{KLMN} = D_{AI}D_{BK}\Omega_{JLMN}$. Since for any tensor X_P

$$[D_{AI}, D_{BK}] X_P = \epsilon_{AB} \epsilon^{ST} \Omega_{PIKS} X_T, \quad (2.7)$$

we get, after commuting the derivatives $D_{AI}D_{BK}$,

$$\begin{aligned} D_{AI}D_{BK}\Omega_{JLMN} &= D_{BK}D_{AI}\Omega_{JLMN} + \epsilon_{AB} \epsilon^{ST} \Omega_{JIKS}\Omega_{LMNT} \\ &\quad + \epsilon_{AB} \epsilon^{ST} (\Omega_{IKLS}\Omega_{JMNT} + \Omega_{IKMS}\Omega_{JNLT} + \Omega_{IKNS}\Omega_{JLMT}). \end{aligned} \quad (2.8)$$

Finally, using eq. (2.5) again as $D_{BK}D_{AI}\Omega_{JLMN} = D_{BK}D_{AL}\Omega_{IJMN}$, we see that

$$\begin{aligned} \epsilon^{AB} D_{AI}D_{BJ}\Omega_{KLMN} &= \epsilon^{AB} D_{BK}D_{AL}\Omega_{IJMN} - 2\epsilon^{ST} \Omega_{JIKS}\Omega_{LMNT} \\ &\quad - 2\epsilon^{ST} (\Omega_{IKLS}\Omega_{JMNT} + \Omega_{IKMS}\Omega_{JNLT} + \Omega_{IKNS}\Omega_{JLMT}). \end{aligned} \quad (2.9)$$

The left hand side of eq. (2.6) is antisymmetric in I, J , while the first two terms in the right hand side of eq. (2.9) are symmetric and the other terms are antisymmetric in these indices. Therefore

$$\begin{aligned} \epsilon^{AB} (D_{AI}D_{BJ} - D_{AJ}D_{BI}) \Omega_{KLMN} \\ = -4\epsilon^{ST} (\Omega_{IKLS}\Omega_{JMNT} + \Omega_{IKMS}\Omega_{JNLT} + \Omega_{IKNS}\Omega_{JLMT}). \end{aligned} \quad (2.11)$$

In verifying explicitly at the end of Section 3 that certain Feynman diagram expressions give three-manifold invariants, (2.11) will play roughly the role that the Jacobi identity for a Lie algebra plays in the analogous calculation in Chern-Simons gauge theory. A Lie algebra \mathcal{J} with invariant quadratic form Tr is defined by a tensor $f \in \Lambda^3 \mathcal{J}$ obeying a Jacobi identity that is schematically $ff + ff + ff = 0$ (where indices are arranged differently in the three terms). In replacing a Lie algebra by a hyper-Kähler manifold, f is replaced by the tensor Ω , and the Bianchi identity by eq. (2.11), which reads schematically $\Omega\Omega + \Omega\Omega + \Omega\Omega =$ total derivative.

2.3 Background in six dimensions

A supersymmetric sigma-model involving maps to a hyper-Kähler manifold X can be defined first of all in six dimensions. A superspace (or supermanifold) construction of this theory is not known, but nevertheless the supersymmetric Lagrangian exists. We describe this for background, though we will not actually work in six dimensions in this paper. In what follows, \mathbf{R}^6 is six-dimensional Euclidean space with a flat metric.

The fields of the six-dimensional supersymmetric model are a map $\Phi : \mathbf{R}^6 \rightarrow X$ (which if we pick local coordinates ϕ^i on X can be described by giving functions ϕ^i on \mathbf{R}^6) and fermions ψ taking values in $\mathcal{S}_+ \otimes \Phi^*(V)$, where \mathcal{S}_+ is one of the spin bundles of \mathbf{R}^6 , and V is the $Sp(n)$ bundle over X that entered in the last section. Recall that \mathcal{S}_+ is a rank four complex bundle. In particular, $\Lambda^4 \mathcal{S}_+$ is one-dimensional, with a Lorentz-invariant generator which we will call ϵ . (This symbol will appear only in the present paragraph, and should cause no confusion with the use of the same name for invariant antisymmetric tensors on other bundles.) If we use Greek letters α, β, \dots for \mathcal{S}_+ -valued objects, then ϵ is a fourth rank antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$. The supersymmetric Lagrangian is

$$L = \int_{\mathbf{R}^6} d^6x \left(\frac{1}{2}(d\Phi, d\Phi) + \frac{i}{2}(\psi, D\psi) + \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\psi_{\alpha I}\psi_{\beta J}\psi_{\gamma K}\psi_{\delta L}\Omega^{IJKL} \right). \quad (2.12)$$

Here D is the Dirac operator mapping $\mathcal{S}_+ \otimes \Phi^*V \rightarrow \mathcal{S}_- \otimes \Phi^*V$, and $(\ , \)$ are the natural metrics on $T^*\mathbf{R}^6 \otimes \Phi^*(TX)$ and $\mathcal{S}_- \otimes \Phi^*V$.

For our purposes, we want to reduce this theory to a dimension less than six in which it can be “twisted” to give a topological field theory. The largest such dimension is three. Reducing the theory to three dimensions means simply restricting to fields that are invariant under a three-dimensional group of translations of \mathbf{R}^6 . For instance, we pick coordinates x^1, x^2, \dots, x^6 on \mathbf{R}^6 , such that the metric is $\sum_i (dx^i)^2$, and we require that the fields be independent of x^4, x^5 , and x^6 . We are then left with a theory on \mathbf{R}^3 with an $SO(3) \times SO(3)$ symmetry, where one $SO(3)$ – acting on x^1, x^2, x^3 – is the rotation symmetry of the three-dimensional theory, and the other – acting on x^4, x^5, x^6 – is an “internal” symmetry. We call these $SO(3)_E$ and $SO(3)_N$, and we call their double covers $SU(2)_E$ and $SU(2)_N$. Allowing for the presence of the fermions, the symmetry group of the theory is actually $(SU(2)_E \times SU(2)_N)/\mathbf{Z}_2$.

The supercharges of the theory transform as two copies of $(\mathbf{2}, \mathbf{2})$ under $SU(2)_E \times SU(2)_N$. Now we “twist” the theory in the following way. We let $SU(2)'$ be a diagonal subgroup of $SU(2)_E \times SU(2)_N$, and we define a new action of rotations by thinking of $SU(2)'$ as the rotation generators. The point of this is that the supercharges transform as two copies of $\mathbf{1} \oplus \mathbf{3}$ under $SU(2)'$. In particular, there are two $SU(2)'$ -invariant supercharges. If we call these Q_A , $A = 1, 2$, then they obey $\{Q_A, Q_B\} = 0$, as one sees by restricting the underlying six-dimensional superalgebra.

Having rotation-invariant supercharges Q_A that square to zero in the twisted theory usually means that the twisted theory can be generalized from flat space to an arbitrary curved manifold (in this case of dimension three) in a way that preserves conservation of the Q_A and such that the metric dependence is of the form $\{Q_A, \dots\}$. When that can be done, then by restricting to the Q_A -invariant observables, one gets a topological field theory.

In the case at hand, this program can be carried out. The topological field theory that one gets is written down in the next section. Note that as the fermions of the untwisted theory transform as $(\mathbf{2}, \mathbf{2})$ of $SU(2)_E \times SU(2)_N$, they transform as $\mathbf{1} \oplus \mathbf{3}$ under $SU(2)'$. In other words, the fermions will be a zero-form η and a one-form χ_μ . Of course, both η and χ_μ take values in $\Phi^*(V)$. The formulas in the next section were found by beginning with the conventional $N = 4$ supersymmetric sigma-model in three dimensions (which can be

obtained as explained above starting from the sigma-model in six dimensions) and rewriting the standard formulas from the twisted point of view. In the next section, we simply write down the result without further commentary.

One important point is that non-linear sigma-models in three dimensions are unrenormalizable and ill-behaved quantum mechanically, and therefore we do *not* claim that the non-topological observables of these theories are naturally defined. In computing the topological observables, it will turn out that everything comes from low order terms that are not sensitive to the unrenormalizability. Therefore, pragmatically, we will not have to commit ourselves to a particular point of view about this issue. One plausible point of view is that by adding higher derivative terms (of the general form $\{Q, \cdot\}$) one could cure the unrenormalizability (at some cost in beauty) without changing the topological observables or our computations of them.

2.4 The topological lagrangian and BRST symmetries

We will now describe the topological sigma-model that can be found using the recipe of the last subsection. We work on an oriented three-dimensional manifold M . We denote local coordinates on M as x^μ , $\mu = 1, 2, 3$. M is endowed with a metric $h_{\mu\nu}$, but an appropriate class of observables of the model will be metric-independent. The target space of the sigma-model is a $4n$ -dimensional hyper-Kähler manifold X . The bosonic scalar fields are a map $\Phi : M \rightarrow X$ which (once local coordinates are chosen on X) can be described via functions $\phi^i(x^\mu)$, $i = 1, \dots, 4n$. The fermions are a scalar η^I and a one-form χ_μ^I with values in V .

The sigma-model action is

$$S = \int_M (L_1 + L_2) \sqrt{h} d^3x, \quad (2.13)$$

with

$$L_1 = \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + \epsilon_{IJ} \chi_\mu^I \nabla^\mu \eta^J, \quad (2.14)$$

$$L_2 = \frac{1}{2} \frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} \left(\epsilon_{IJ} \chi_\mu^I \nabla_\nu \chi_\rho^J + \frac{1}{3} \Omega_{IJKL} \chi_\mu^I \chi_\nu^J \chi_\rho^K \eta^L \right), \quad (2.15)$$

The covariant derivative of fermions, here denoted as ∇_μ , is defined using the Levi-Civita connection on M and the pullback of the Levi-Civita connection on V

$$\nabla_\mu = \partial_\mu \delta^I{}_J + (\partial_\mu \phi^i) \Gamma_{iJ}^I. \quad (2.16)$$

The Lagrangians L_1 and L_2 are each separately invariant under a pair of BRST symmetries Q_A . A transformation $\varepsilon^A Q_A$ acts on the fields according to the formula

$$\delta_\varepsilon \phi^i = \gamma_{AI}^i \varepsilon^A \eta^I, \quad (2.17)$$

$$\delta_\varepsilon \eta^I = 0, \quad (2.18)$$

$$\delta_\varepsilon \chi_\mu^I = \varepsilon^A \epsilon_{AB} \gamma_i^{BI} \partial_\mu \phi^i - \Gamma_{iJ}^I (\delta_\varepsilon \phi^i) \chi_\mu^J. \quad (2.19)$$

The Q_A obey

$$\{Q_A, Q_B\} = 0, \quad (2.20)$$

as promised in the last subsection.

If among the infinitely many complex structures of the hyper-Kähler manifold X (which are parametrized by a two-sphere) one picks a particular one, and one lets ϕ^I be local holomorphic coordinates in this complex structure, then one can pick a basis Q, \bar{Q} for the two supercharges in which they act by

$$\begin{aligned} \delta \phi^I &= \eta^I, & \delta \bar{\phi}^{\bar{I}} &= 0, \\ \delta \eta^I &= 0, & \delta \chi_\mu^I &= \epsilon^{IJ} g_{J\bar{K}} \partial_\mu \bar{\phi}^{\bar{K}} - \Gamma_{JK}^I \eta^J \chi_\mu^K \end{aligned} \quad (2.21)$$

for Q and

$$\begin{aligned} \delta \phi^I &= 0, & \delta \bar{\phi}^{\bar{I}} &= g^{\bar{I}J} \epsilon_{JK} \eta^K, \\ \delta \eta^I &= 0, & \delta \chi_\mu^I &= -\partial_\mu \phi^I \end{aligned} \quad (2.22)$$

for \bar{Q} .

Now, let us verify formally that this theory is a topological field theory. L_2 is manifestly independent of the metric of M ; it is written just in terms of wedge products and exterior

derivatives of differential forms. L_1 is metric-dependent, but can be written as $\{Q_A, \cdot\}$ for any choice of A . In fact,

$$\{\epsilon^A Q_A, \epsilon_{IJ} \gamma_i^{BI} \chi_\mu^J \partial^\mu \phi^i\} = \epsilon^B L_1. \quad (2.23)$$

Since the metric-dependence of S is thus of the form $\{Q, \cdot\}$, the partition function and more generally the correlation functions of Q -invariant operators are metric independent.

2.5 Topological observables

Now we want to introduce some topological observables. These are the operators which (anti)commute with Q_A but can not be presented as (anti)commutators of Q_A with some other operators. The vacuum matrix elements of such operators are topological invariants of the manifold M .

The simplest topological operators can be constructed from closed forms on X . Once one picks a particular complex structure on X , the tensor $\epsilon^{IJ} g_{J\bar{K}}$ establishes an isomorphism between the space $\Omega^{l,0}$ of $(l,0)$ -forms and the space $\Omega^{0,l}$ of $(0,l)$ -forms on X . Let ω be a l -form which is ∂ -closed as a $(l,0)$ -form and $\bar{\partial}$ -closed as a $(0,l)$ -form. Then a similarity between the action (2.21), (2.22) of the BRST operators Q, \bar{Q} on the pairs of fields (ϕ^I, η^I) and $(\bar{\phi}^{\bar{I}}, \eta^{\bar{I}})$, and the action of the operators $\partial, \bar{\partial}$ on the differential forms on X allows us to conclude that the operator

$$\mathcal{O}_\eta(\omega) = \omega_{I_1 \dots I_l} \eta^{I_1} \dots \eta^{I_l} \quad (2.24)$$

is BRST-closed. These operators are further discussed in Section 5.

A more subtle topological operator can be constructed with the help of the pullback of the connection Γ_{iJ}^I from the $Sp(n)$ bundle V over X via the map $\Phi : M \rightarrow X$. This pullback $\partial_\mu \phi^i \Gamma_{iJ}^I$ is not Q -invariant, and so cannot be used to define knot invariants. However, it is possible to modify it in order to achieve Q -invariance. Indeed, the modified connection

$$A_{\mu IJ} = \partial_\mu \phi^i \epsilon_{IK} \Gamma_{iJ}^K + \Omega_{IJKL} \chi_\mu^K \eta^L \quad (2.25)$$

is readily seen to be Q -invariant up to a gauge transformation. Hence if \mathcal{K} is a knot in M and \mathcal{V}_α is a representation of $Sp(n)$, then the operator

$$\mathcal{O}_\alpha(\mathcal{K}) = \text{Tr}_\alpha \text{Pexp} \left(\oint_{\mathcal{K}} A_{\mu IJ} dx^\mu \right), \quad (2.26)$$

which is the trace of the holonomy of $A_{\mu IJ}$ along \mathcal{K} taken in the representation \mathcal{V}_α , is Q -invariant and independent of the metric on M . The expectation value of the product of such operators should give invariants of framed links, rather as in Chern-Simons gauge theory. For example, for knots in S^3 , one would expect to obtain knot invariants of finite type, generalizing those obtained from the gauge theory.

2.6 Sigma-model and Chern-Simons theory

It may now be time to begin to disclose the secret of how to think about this theory. L_1 and L_2 play a completely different role. The fact that L_2 is metric-independent suggests that it is more fundamental, so let us think about it first.

L_2 has a structure very reminiscent of Chern-Simons theory, with the one-form χ playing the role of the connection A of Chern-Simons theory. Indeed, compare L_2 , schematically $\chi \wedge D\chi + \chi \wedge \chi \wedge \chi \eta$, to the Lagrangian of Chern-Simons theory, schematically $A \wedge DA + A \wedge A \wedge A$ (D is here the covariant derivative with respect to a background flat connection about which we may be expanding). It is true that in L_2 the interaction term is not just χ^3 but has an extra factor of η . However, as we will see, this does not spoil the analogy³.

In quantizing Chern-Simons theory, one must introduce, in addition to A , certain extra fields: anticommuting ghosts and antighosts c and \bar{c} and a commuting scalar auxiliary field t (sometimes called the Nielsen-Lautrup auxiliary field), all in the adjoint representation of the group.⁴ This theory usually is quantized with just one fermionic (BRST) symmetry, the transformation laws being $\delta \bar{c} = t$, $\delta t = 0$, and $\delta c^a = f_{bc}^a c^b c^c$, $\delta A_\mu = -D_\mu c$, with f the structure constants of G .

³In fact, η plays the role of a “fermionic coupling constant,” making sure that we do not have to go too far in perturbative expansions.

⁴In some gauges, t can be eliminated while preserving locality.

Let us think of our topological sigma-model as a sort of Chern-Simons theory with the statistics reversed. We make the following identifications: η corresponds to t , and after picking one of the complex structures on X , \bar{c} corresponds to the local antiholomorphic coordinates $\bar{\phi}^I$, and c to the local holomorphic coordinates ϕ^I . A peek back to the \bar{Q} transformation laws of the sigma-model shows that they match very nicely with the BRST transformation laws of gauge-fixed Chern-Simons theory, as written in the last paragraph.

Moreover, the analogy can be pursued further. To quantize Chern-Simons theory, one must add a gauge fixing term of the form $\{Q, \cdot\}$. Not only is L_1 likewise of the form $\{\bar{Q}, \cdot\}$, but the match is much closer. The most common gauge fixing term of Chern-Simons theory is schematically $\Delta L = \partial_\mu c \partial^\mu \bar{c} + t D_\mu A^\mu$. If we replace t by η , A by χ , $\bar{\phi}$ by \bar{c} , and ϕ by c , ΔL does match up nicely with the detailed form of L_1 . The topological observable (2.26) corresponds to the holonomy operator $\text{Tr Pexp}(\oint_{\mathcal{K}} A)$ of the Chern-Simons theory which produces the colored Jones polynomial. Note that the role of the Chern-Simons gauge field A is played in (2.26) by the modifying term $\Omega \chi \eta$ of eq. (2.25). (We will later see that the “ordinary” pullback term can be dropped.)

So our course is set. We will try to compare the topological sigma model to Chern-Simons theory, with the curvature tensor Ω_{IJKL} of the hyper-Kähler manifold playing the role of the structure constants f_{abc} of the Chern-Simons gauge group. Of course, f is antisymmetric while Ω is symmetric. This is what comes of reversing the statistics of the gauge field. Moreover, f_{abc} has three indices while Ω_{IJKL} has four. This actually will lead to the main difference between the two theories. The extra index of Ω is coupled to an η field, which will be absorbed by an η zero mode. (Other contributions will be seen to vanish.) As η has a definite number of zero modes, only diagrams with a definite number of vertices will contribute. That is why the topological sigma-model will be related to three-manifold invariants associated with trivalent graphs with a definite number of loops – in fact $n+1$ loops if X has dimension $4n$. Such a restriction on the number of loops, which makes the theory vastly more elementary, would be impossible in a unitary field theory such as Chern-Simons theory, but the twisting that made the sigma-model topological has spoiled unitarity.

Anticipating the analogy with Chern-Simons theory will greatly facilitate the analysis of the next section.

3 Perturbative calculations

Consider the partition function of the topological sigma-model

$$Z(M) = \int \mathcal{D}\phi^i \mathcal{D}\eta^I \mathcal{D}\chi_\mu^I \exp \left(- \int (c_1 L_1 + c_2 L_2) \sqrt{\hbar} d^3 x \right). \quad (3.1)$$

Here $L_{1,2}$ are the lagrangians (2.14) and (2.15), and $c_{1,2}$ are some arbitrary constants. Since L_1 is BRST-exact, the partition function does not depend on c_1 . A field rescaling

$$\eta^I \rightarrow \lambda \eta^I, \quad \chi_\mu^I \rightarrow \lambda^{-1} \chi_\mu^I \quad (3.2)$$

does not change c_1 but changes c_2 : $c_2 \rightarrow \lambda^{-2} c_2$. Thus $Z(M)$ does not depend on c_2 either.⁵ We choose $c_1 = c_2 = \frac{1}{\hbar}$ so that

$$Z(M) = \int \mathcal{D}\phi^i \mathcal{D}\eta^I \mathcal{D}\chi_\mu^I \exp \left(- \frac{1}{\hbar} S \right), \quad (3.3)$$

where S is the action (2.13). The partition function (3.3) should be independent of the value of \hbar .

For small \hbar , some form of perturbative expansion should be valid. The minima of the action are the constant maps of M to X , so we will expand around those. One must as usual be careful with bosonic and fermionic zero modes. The bosonic zero modes are simply the constant modes of ϕ – displacements in the constant map of M to X . In expanding around a constant map of M to X , the fermionic zero modes, if M is a rational homology sphere (i.e. if the first Betti number $b_1 = 0$), are the constant modes of η ; they are equal in number to the number of components of η , which is $2n$ if X is of dimension $4n$. If M is not a rational homology sphere but has first Betti number $b_1 > 0$, then there are in addition $2nb_1$ zero modes of χ_μ .

⁵This argument is valid for Feynman diagrams of more than one loop, but not for the one-loop determinants, which in any case will be examined closely below.

To take account of the bosonic zero modes in a perturbative expansion, one must introduce “collective coordinates,” and integrate over the space of all constant maps of M to X . Thus, we split the bosonic field $\phi^i(x)$ into a sum of a constant and fluctuating part,

$$\phi^i(x) = \varphi_0^i + \varphi^i(x), \quad (3.4)$$

where φ_0^i is constant, and $\varphi^i(x)$ is required to be orthogonal to the zero mode. The zero modes of $\phi^i(x)$ are thus contained in the constant part φ_0^i . We define a partition function $Z_X(M; \varphi_0^i)$ of fixed φ_0^i , and obtain the partition function $Z_X(M)$ as an integral over X

$$Z_X(M) = \frac{1}{(2\pi\hbar)^{2n}} \int_X Z(M; \varphi_0^i) \sqrt{g} d^{4n} \varphi_0^i. \quad (3.5)$$

The perturbative calculation presents the integrand $Z(M; \varphi_0^i)$ of (3.5) as a product of two factors

$$Z(M; \varphi_0^i) = Z_0(M; \varphi_0^i) Z_{\eta\chi\varphi}(M, X; \varphi_0^i). \quad (3.6)$$

Here $Z_0(M; \varphi_0^i)$ is the 1-loop contribution of non-zero modes, while $Z_{\eta\chi\varphi}(M, X; \varphi_0^i)$ is the exponential of the sum of all Feynman diagrams of two or more loops, in the background field of given φ_0 .

3.1 The one-loop contribution

Let us first determine the one-loop contribution $Z_0(M; \varphi_0^i)$. We work with the part of the action (2.13) which is quadratic in fluctuating bosonic fields $\varphi^i(x)$ and in fermionic fields η^I, χ_μ^I :

$$S_0 = \int_M \sqrt{h} d^3x \left(\frac{1}{2} g_{ij} \partial_\mu \varphi^i \partial^\mu \varphi^j + \epsilon_{IJ} \chi_\mu^I \nabla^\mu \eta^J + \frac{1}{2} \frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} \epsilon_{IJ} \chi_\mu^I \nabla_\nu \chi_\rho^J \right) \quad (3.7)$$

(the tensors g_{ij} , ϵ_{IJ} and implicit Christoffel symbols Γ_{jk}^i in this formula are taken at the point φ_0^i of X). The path integral of the bosonic fields φ^i gives a factor

$$\left(\det' \Delta_{(0)} \right)^{-2n}, \quad (3.8)$$

where $\Delta_{(0)}$ is the Laplacian acting on zero-forms on M and \det' means that we exclude the (constant) zero modes.

Now introduce an operator L_- which acts on the direct sum of zero- and one-forms on M :

$$L_-(\eta, \chi_\mu) = (-\nabla^\mu \chi_\mu, \nabla_\mu \eta + h_{\mu\nu} \frac{1}{\sqrt{h}} \epsilon^{\nu\rho\lambda} \partial_\rho \chi_\lambda). \quad (3.9)$$

If we define a scalar product

$$\langle \eta, \chi_\mu | \eta', \chi'_\mu \rangle = \int_M \sqrt{h} d^3x (\eta \eta' + h^{\mu\nu} \chi_\mu \chi'_\nu), \quad (3.10)$$

then the fermionic part of the action (3.7) becomes a quadratic form

$$\frac{1}{2} \epsilon_{IJ} \langle \eta^I, \chi_\mu^I | L_- | \eta^J, \chi_\mu^J \rangle. \quad (3.11)$$

We assume that the lattice-regularized expression for the fermionic integration measure of eq. (3.1) is

$$\mathcal{D}\eta^I \mathcal{D}\chi_\mu^I = \prod_{x \in M} \frac{\hbar^n}{n!} \left(\epsilon_{IJ} d\eta^I(x) d\eta^J(x) \right)^n \prod_{x \in M} \frac{\hbar^{3n}}{(3n)!} \left(\epsilon_{IJ} h^{\mu\nu} d\chi_\mu^I(x) d\chi_\nu^J(x) \right)^{3n}, \quad (3.12)$$

here $\prod_{x \in M}$ means a product over all nodes of the lattice which approximates M . For this choice of integration measure, the fermionic one-loop contribution, with zero modes removed, is

$$(\det' L_-)^n, \quad (3.13)$$

so that the total one-loop contribution of non-zero modes is

$$Z_0(M; \varphi_0^i) = \left[\frac{\det' L_-}{(\det' \Delta_0)^2} \right]^n. \quad (3.14)$$

Note in particular that $Z_0(M; \varphi_0)$ in its form (3.14) is independent of φ_0 , therefore we may denote it simply as $Z_0(M)$.

The fermionic zero modes are the zero modes of the operator L_- . More precisely, the zero modes of η^I and χ_μ^I are harmonic 0- and 1-forms on M tensored with the fiber V_{φ_0} of

the $Sp(n)$ bundle $V \rightarrow X$ at the point φ_0 . Therefore, as anticipated above,

$$\#(\text{zero modes of } \eta^I) = 2n, \quad (3.15)$$

$$\#(\text{zero modes of } \chi_\mu^I) = 2nb_1. \quad (3.16)$$

Because of the fermion zero modes, a product of $2n$ fields η^I and $2nb_1$ fields χ_μ^I has a non-zero vacuum expectation value. The products of η^I and χ_μ^I zero modes are elements of the spaces

$$H_\eta = \Lambda^{\max} \left(H^0(M, \mathbf{R}) \otimes V_{\varphi_0} \right), \quad (3.17)$$

$$H_\chi = \Lambda^{\max} \left(H^1(M, \mathbf{R}) \otimes V_{\varphi_0} \right) \quad (3.18)$$

respectively.

The absolute value of the ratio $\left| \det' L_- / (\det' \Delta_0)^2 \right|$ is equal to the analytic Ray-Singer torsion of (the trivial connection on) M . Therefore its n th power $Z_0(M; \varphi_0^i)$ is not just a number but an element in the top exterior power of the space $(H_0(M, \mathbf{R}) \oplus H^1(M, \mathbf{R})) \otimes V_{\varphi_0}$, or, equivalently, in the top exterior power of the space

$$\left(2H_0(M, \mathbf{R}) \oplus H^0(M, \mathbf{R}) \oplus H^1(M, \mathbf{R}) \right) \otimes V_{\varphi_0}. \quad (3.19)$$

Here $2H_0(M, \mathbf{R})$ denotes $H_0(M, \mathbf{R}) \oplus H_0(M, \mathbf{R})$. An element of the space

$$\Lambda^{\max} (2H_0(M, \mathbf{R}) \otimes V_{\varphi_0})$$

is a volume form for the integral over the bosonic collective coordinates φ_0 , which range over X . Therefore $Z_0(M; \varphi_0^i)$ determines the product of the volume form on X and the vacuum expectation values of the fermionic fields.

There are natural volume forms – the Riemannian volume form of X , and a measure for the fermions that will be written presently. $Z_0(M; \varphi_0^i)$ is the product of the natural volume forms times a number equal to $|H_1(M, \mathbf{Z})|'$, the number of points of finite order (i.e. torsion elements) in $H_1(M, \mathbf{Z})$ (see [7] and references therein). Note that $|H_1(M, \mathbf{Z})|'$ differs from the order of the first homology $|H_1(M, \mathbf{Z})|$, which is usually defined as follows:

$$|H_1(M, \mathbf{Z})| = \begin{cases} \# \text{ elements in } H_1(M, \mathbf{Z}) & \text{if } b_1 = 0 \\ 0 & \text{if } b_1 > 0. \end{cases} \quad (3.20)$$

If M is a rational homology sphere (i.e. $b_1 = 0$) then, obviously, $|H_1(M, \mathbf{Z})|' = |H_1(M, \mathbf{Z})|$.

The natural volume forms (which must be multiplied by $|H_1(M, \mathbf{Z})|'$ to get $Z_0(M)$) are as follows. There is a lattice inside $H^1(M, \mathbf{R})$ which is formed by one-forms with integer integrals over one-cycles of M . Let $\omega_\mu^{(\alpha)}$, $1 \leq \alpha \leq b_1$ be harmonic one-forms forming a basis of this lattice. The natural measures for the fermion zero modes can be described by giving the fermionic vacuum expectation values

$$\langle \eta^{I_1}(x_1) \cdots \eta^{I_{2n}}(x_{2n}) \rangle = \hbar^n \epsilon^{I_1 \cdots I_{2n}}, \quad (3.21)$$

$$\langle \chi_{\mu_1}^{I_1}(x_1) \cdots \chi_{\mu_{2nb_1}}^{I_{2nb_1}}(x_{b_1}) \rangle = \frac{\hbar^{nb_1}}{((2n)!)^{b_1}} \sum_{s \in S_{2nb_1}} (-1)^{|s|} \quad (3.22)$$

$$\times \prod_{\alpha=0}^{b_1-1} \left(\epsilon^{I_{s(2\alpha n+1)} \cdots I_{s(2\alpha n+2n)}} \omega_{\mu_{s(2\alpha n+1)}}^{(\alpha)}(x_{s(2\alpha n+1)}) \cdots \omega_{\mu_{s(2\alpha n+2n)}}^{(\alpha)}(x_{s(2\alpha n+2n)}) \right)$$

in calculating the Feynman diagrams which absorb the zero modes. We used the following notation in eqs. (3.21) and (3.22):

$$\epsilon^{I_1 \cdots I_{2n}} = \frac{1}{(2n)!} \sum_{s \in S_{2n}} (-1)^{|s|} \epsilon^{I_{s(1)} I_{s(2)} \cdots I_{s(2n-1)} I_{s(2n)}}, \quad (3.23)$$

S_m is the symmetric group of m elements, and $|s|$ is the parity of a permutation s .

A choice of an overall sign in the formulas (3.21) and (3.22) for the fermionic expectation values is equivalent to a choice of orientations on the spaces

$$H^0(M, \mathbf{R}) \otimes V_{\varphi_0}, \quad H^1(M, \mathbf{R}) \otimes V_{\varphi_0}. \quad (3.24)$$

As a result, the sign of the whole partition function $Z_X(M)$ depends on this choice. In other words, at a first glance, $Z_X(M)$ is an invariant of M with the choice of orientation on the spaces (3.24), rather than simply an invariant of M . An orientation of the space V_{φ_0} is determined by the n th power of the two-form ϵ_{IJ} on X . Since the space V_{φ_0} is even-dimensional, the orientation on the spaces (3.24) does not depend on the choice of orientation on the cohomology spaces $H^0(M, \mathbf{R})$, $H^1(M, \mathbf{R})$ (this is why the sign of the expectation value (3.22) does not depend on the choice of basic one-forms $\omega_\mu^{(\alpha)}$). Therefore, the choice of orientation of the spaces (3.24) and, consequently, the choice of the sign in eqs. (3.21), (3.22) (which was

induced by the fermionic integration measure (3.12)) is canonical and the invariant $Z_X(M)$ can always be reduced to a canonical orientation.

This situation is similar to the framing anomaly of the quantum Chern-Simons invariant [1]. The Chern-Simons path integral is a topological invariant of framed 3-manifolds; however, since 3-manifolds have canonical 2-framings [23], it can be converted into a genuine invariant of 3-manifolds. Still, the framing anomaly shows up as a framing correction in the surgery formula for the quantum Chern-Simons invariant. We will see in subsection 5.2 that a similar sign correction is required for the surgery formula of $Z_X(M)$.

3.2 Minimal Feynman diagrams

Now we will start to analyze the Feynman diagrams that absorb the fermionic zero modes. We may limit our attention to only those Feynman diagrams (we call them “minimal”) whose contribution is of order \hbar^{2n} . Indeed, the integral (3.5) related to $4n$ bosonic zero modes carries a normalization factor $(2\pi\hbar)^{-2n}$. However, as we have mentioned, due to the topological nature of the sigma-model, the partition function $Z_X(M)$ should not depend on \hbar . Therefore the contribution of all “non-minimal” Feynman diagrams cancels out. Since there are always exactly $2n$ zero modes of η , the vanishing of Feynman diagrams proportional to \hbar^s with $s < 2n$ will be fairly obvious in what follows. Less obvious in direct inspection of the diagrams is the fact that the diagrams with $s > 2n$ also do not contribute.

We want to determine which diagrams are minimal. Consider a diagram with V vertices. Let L be the total number of legs emanating from all of the various vertices. Some of the legs are joined together by propagators, others are attached to fermionic zero modes. Each vertex in the diagram carries a factor of \hbar^{-1} , so the vertices taken together bring a factor of \hbar^{-V} . Each propagator carries a factor of \hbar , and each fermionic zero mode carries a normalization factor of $\hbar^{\frac{1}{2}}$. Therefore the total contribution of propagators and external legs is $\hbar^{\frac{L}{2}}$. So if the diagram is minimal, then

$$\frac{L}{2} - V = 2n. \tag{3.25}$$

The interaction vertices coming from the Lagrangians (2.14) and (2.15) are – in expanding around a constant map to X – of fourth order and above, so

$$L \geq 4V. \quad (3.26)$$

η only appears linearly in the Lagrangian, so all vertices are at most linear in η^I . Therefore if a diagram contains sufficient vertices to absorb all $2n$ zero modes of η^I , then

$$V \geq 2n. \quad (3.27)$$

The conditions (3.25), (3.26) and (3.27) are compatible only if $L = 4V$ and $V = 2n$. We are thus reduced to a finite set of diagrams. Moreover, the knowledge that the minimal diagrams should contain only fourth order vertices which are proportional to η^I leads to a drastic simplification in the effective Lagrangian. The action (2.13) contains two vertices with the requisite properties

$$V_1 = \frac{1}{6} \frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} \Omega_{IJKL} \chi_\mu^I \chi_\nu^J \chi_\rho^K \eta^L, \quad (3.28)$$

$$V_2 = -\frac{1}{2} \gamma_i^{AK} \gamma_j^{BL} \epsilon_{AB} \Omega_{IJKL} \chi_\mu^I \eta^J (\partial^\mu \varphi^i) \varphi^j. \quad (3.29)$$

The vertex V_1 comes directly from L_2 of eq. (2.15), while V_2 comes from the expansion of the connection Γ_{iJ}^I , which appears in the covariant derivative ∇^μ in the second term of the right hand side of eq. (2.14), in powers of φ^i . Thus for the purpose of calculating the minimal diagrams in our sigma-model, we can use the “minimal” Lagrangian

$$\begin{aligned} L_{\min} = & \frac{1}{2} g_{ij} \partial_\mu \varphi^i \partial^\mu \varphi^j + \epsilon_{IJ} \chi_\mu^I \partial^\mu \eta^J + \frac{1}{2} \frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} \epsilon_{IJ} \chi_\mu^I \partial_\nu \chi_\rho^J \\ & + \frac{1}{6} \frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} \Omega_{IJKL} \chi_\mu^I \chi_\nu^J \chi_\rho^K \eta^L - \frac{1}{2} \gamma_i^{AK} \gamma_j^{BL} \epsilon_{AB} \Omega_{IJKL} \chi_\mu^I \eta^J (\partial^\mu \varphi^i) \varphi^j. \end{aligned} \quad (3.30)$$

The analogy – described at the end of the last section – between this theory and Chern-Simons theory can now be considerably perfected. Instead of a non-polynomial sigma-model action, we now have reduced the discussion to a polynomial action. Moreover, though the vertices are quartic, each vertex is linear in η . Since we have $2n$ zero modes of η and will be looking at diagrams with precisely $2n$ vertices, η will never appear in propagators; all factors

of η in vertices will go immediately into absorbing zero modes. What is more, as the zero mode wave functions are constant, the absorption of the extra η zero mode at each vertex will add no additional complications to the integrals over M associated with the diagrams. After thus eliminating the η 's from the vertices, we reduce to a theory in which the vertices are all cubic, just as in Chern-Simons theory. Using the dictionary comparing the sigma-model to Chern-Simons theory that was described at the end of the last section, it can further be seen that the vertices in the sigma-model have just the same structure as the vertices (namely $A \wedge A \wedge A$ and $\bar{c} A_i D^i c$) of gauge-fixed Chern-Simons theory. The diagrams of this theory will thus coincide with the diagrams of Chern-Simons theory, but the weight factors are different, since the vertices given in the last paragraph are proportional to the curvature tensor Ω of X , rather than the structure constants of a Lie group.

3.3 Propagators

Each minimal diagram contains a total of $2n$ vertices (3.28) and (3.29). All fields η in these vertices are used to absorb the zero modes, so we need to know only the propagators $\langle \varphi^i(x_1) \varphi^j(x_2) \rangle$ and $\langle \chi_\mu^I(x_1) \chi_\nu^J(x_2) \rangle$. These propagators can be expressed in terms of Green's functions $G^{(\varphi)}(x_1, x_2)$ and $G_{\mu\nu}^{(x)}(x_1, x_2)$. $G^{(\varphi)}(x_1, x_2)$ is a symmetric function which satisfies the equation

$$\Delta^{(1)} G^{(\varphi)}(x_1, x_2) = \delta(x_1 - x_2) - \frac{1}{\text{Vol}(M)}. \quad (3.31)$$

Here $\Delta^{(1)}$ is a covariant Laplacian acting on the first argument x_1 , $\text{Vol}(M)$ is the volume of M ,

$$\text{Vol}(M) = \int_M \sqrt{h} d^3x, \quad (3.32)$$

and the δ -function is normalized by a condition

$$\int_M \delta(x_1 - x_2) f(x_2) \sqrt{h(x_2)} d^3x_2 = f(x_1). \quad (3.33)$$

$G_{\mu\nu}^{(x)}(x_1, x_2)$ is a symmetric one-form in both of its arguments. It is co-closed and has the following property: if $\mathcal{K}_1, \mathcal{K}_2$ are two knots in M , then their linking number can be calculated

as a double integral

$$\oint_{\mathcal{K}_1} dx_1^{\mu_1} \oint_{\mathcal{K}_2} dx_2^{\mu_2} G_{\mu_1 \mu_2}^{(\chi)}(x_1, x_2). \quad (3.34)$$

A more formal way to define $G_{\mu\nu}^{(\chi)}(x_1, x_2)$ is to say that there exists a function $\tilde{G}_{\mu\nu}(x_1, x_2)$ which is a two-form in x_1 and a zero-form in x_2 , such that

$$\frac{1}{\sqrt{h(x_1)}} \epsilon^{\mu\nu\rho} \frac{\partial}{\partial x_1^\nu} G_{\rho\lambda}^{(\chi)}(x_1, x_2) = \delta_\lambda^\mu \delta(x_1 - x_2) + \frac{1}{\sqrt{h(x_1)}} \epsilon^{\mu\nu\rho} \frac{\partial}{\partial x_2^\lambda} \tilde{G}_{\nu\rho}(x_1, x_2). \quad (3.35)$$

$G_{\mu\nu}^{(\chi)}(x_1, x_2)$ also satisfies a condition

$$\frac{\partial}{\partial x_1^\mu} \left(\sqrt{h(x_1)} h^{\mu\nu}(x_1) G_{\nu\rho}^{(\chi)}(x_1, x_2) \right) = 0. \quad (3.36)$$

The propagators of φ^i and χ_μ^I are

$$\langle \varphi^i(x_1) \varphi^j(x_2) \rangle = -\hbar g^{ij} G^{(\varphi)}(x_1, x_2), \quad (3.37)$$

$$\langle \chi_\mu^I(x_1) \chi_\nu^J(x_2) \rangle = \hbar \epsilon^{IJ} G_{\mu\nu}^{(\chi)}(x_1, x_2). \quad (3.38)$$

3.4 Feynman graphs and weight functions

It is useful for the future discussion to introduce a notion of a *Feynman graph* corresponding to a given Feynman diagram. The Feynman graph is obtained by removing the legs of the Feynman diagram which absorb the fermionic zero modes, and by ignoring the difference between the $\varphi\varphi$ and $\chi\chi$ propagators.

The types of Feynman graphs participating in the calculation of $Z_{\eta\chi\varphi}(M, X; \varphi_0^i)$ depend only on the dimension of the hyper-Kähler manifold X and on the first Betti number b_1 of the 3-manifold M . X determines the number of vertices in the graphs, which is equal to $2n$.

If M is a rational homology sphere (i.e. if $b_1(M) = 0$) then there are no χ zero modes to absorb. As a result, all χ fields in the Feynman diagrams are connected by propagators and the corresponding Feynman graphs are closed graphs with $2n$ tri-valent vertices. The same type of graphs appeared in the calculation of the trivial connection contribution to the quantum Chern-Simons invariant of rational homology spheres.

If $b_1(M) = 1$ then there are $2n$ zero modes of χ coming from a harmonic one-form $\omega_\mu^{(1)}$ on M . Since $\omega^{(1)} \wedge \omega^{(1)} = 0$, each vertex in a minimal Feynman diagram can absorb at most one zero mode of χ . Therefore each vertex absorbs exactly one zero mode of χ , and a corresponding Feynman graph is a closed graph with $2n$ bi-valent vertices. Obviously, such graphs are collections of circles with vertex insertions. They appear also in the Chern-Simons calculation of the Reidemeister torsion in the background of a $U(1)$ connection proportional to the one-form $\omega_\mu^{(1)}$.

If $b_1(M) = 2$, then there are $4n$ zero modes of χ coming from two harmonic one-forms $\omega^{(1)}, \omega^{(2)}$ on M . Each vertex in a minimal Feynman diagram can absorb at most one zero mode of χ proportional to a given one-form. Therefore, each vertex should absorb exactly two zero modes of χ , one coming from $\omega^{(1)}$ and the other coming from $\omega^{(2)}$. Since the vertex (3.29) has only one field χ , it can not participate in the Feynman diagrams when $b_1(M) > 1$. The Feynman graphs have $2n$ uni-valent vertices, so they look like a collection of n segments. For reasons we will explain later, the integral of a product of a $\chi\chi$ propagator and one-forms $\omega^{(1)}, \omega^{(2)}$ which corresponds to each segment, is proportional to a Massey product or fourth order “Milnor linking number” of the forms $\omega^{(1)}$ and $\omega^{(2)}$.

If $b_1(M) = 3$, then there are $6n$ zero modes of χ which are proportional to three harmonic one-forms $\omega^{(1)}, \omega^{(2)}, \omega^{(3)}$ on M . Each vertex in the Feynman diagrams absorbs three zero modes of χ (one one-form of each kind). The Feynman graph is a collection of $2n$ totally disconnected vertices with no edges. Associated to each vertex is an integral $\int_M \omega^{(1)} \wedge \omega^{(2)} \wedge \omega^{(3)}$ which measures the invariant information that is contained in the cubic intersection form on $H^1(M, \mathbf{Z})$. For $b_1(M) = 3$, the M -dependence of the partition function comes entirely through this integral.

If $b_1(M) > 3$, then no minimal Feynman diagram can absorb all zero modes of χ , and the whole partition function $Z_X(M)$ vanishes.

Denote the set of all closed graphs with $2n$ vertices, each of which is m -valent, as $\Gamma_{n,m}$. Let $Z_\Gamma(M, X; \varphi_0^i)$ be the sum of contributions of minimal Feynman diagrams which correspond to a graph Γ . Then the total contribution of Feynman diagrams is a sum over all Feynman

graphs permitted for a given pair of a hyper-Kähler manifold X and a 3-manifold M :

$$Z_{\eta_X \varphi}(M, X; \varphi_0^i) = \sum_{\Gamma \in \Gamma_{n, 3-b_1(M)}} Z_{\Gamma}(M, X; \varphi_0^i). \quad (3.39)$$

Each contribution $Z_{\Gamma}(M, X; \varphi_0^i)$ can be presented as a product of two factors

$$Z_{\Gamma}(M, X; \varphi_0^i) = W_{\Gamma}(X; \varphi_0^i) \sum_a I_{\Gamma, a}(M). \quad (3.40)$$

In this formula $I_{\Gamma, a}(M)$ are the integrals over M of the products of propagators $G^{(\varphi)}(x_1, x_2)$, $G_{\mu\nu}^{(\chi)}(x_1, x_2)$ as well as of the zero modes $\omega_{\mu}^{(\alpha)}(x)$ coming from eq. (3.22). The sum \sum_a goes over all possible ways to assign the vertices (3.37) and (3.38) to the vertices of the Feynman graph Γ . In other words, this sum reflects the fact that for $b_1(M) = 0, 1$ different Feynman diagrams may correspond to the same Feynman graph.

The factor $W_{\Gamma}(X; \varphi_0^i)$ is a product of tensors Ω_{IJKL} coming from the vertices V_1 and V_2 . Their indices are contracted by the tensors ϵ^{IJ} contained in the propagators, and by the tensors $\epsilon^{I_1 \dots I_{2n}}$ contained in the zero mode expectation values (3.21) and (3.22). $W_{\Gamma}(X; \varphi_0^i)$ also includes a sign factor coming from rearranging the fermionic fields in the correlators.

The following technical paragraph contains a precise prescription of how to calculate $W_{\Gamma}(X; \varphi_0^i)$ for a closed graph Γ with $2n$ trivalent vertices. We assign the numbers $1, \dots, 2n$ to the vertices of the graph Γ and the numbers $1, 2, 3$ to the legs of each vertex in the counterclockwise order. We also assign the numbers $1, \dots, 3n$ to the edges of the graph and the numbers $1, 2$ to the endpoints of each edge. Now the graph Γ defines a map σ from the pairs (k, l) , $1 \leq k \leq 3n$, $l = 1, 2$ to the pairs (i, j) , $1 \leq i \leq 2n$, $j = 1, 2, 3$ which describes how the edges are attached to the legs of the vertices. Let s be a permutation which maps a set of pairs

$$(1, 0), (1, 1), (1, 2), (1, 3), \dots, (2n, 0), (2n, 1), (2n, 2), (2n, 3)$$

onto a set

$$(1, 0), \dots, (2n, 0), \sigma(1, 1), \sigma(1, 2), \dots, \sigma(2n, 1), \sigma(2n, 2).$$

We denote by $|s|$ a parity of s . The function $W_{\Gamma}(X; \varphi_0^i)$ is defined as a product of tensors

$$W_{\Gamma}(X; \varphi_0^i) = (-1)^{|s|} \epsilon^{I_{1,0} \dots I_{2n,0}} \prod_{k=1}^{3n} \epsilon^{I_{\sigma(k,1)} I_{\sigma(k,2)}} \prod_{l=1}^{2n} \Omega_{I_{l,0} I_{l,1} I_{l,2} I_{l,3}}. \quad (3.41)$$

Here we assume summation over repeated indices.

$I_{\Gamma,a}$ depends in an interesting way on M but only rather trivially on X : X enters only through its dimension $4n$, which was seen above to determine the numbers V and L of vertices and legs in our diagrams. Conversely, $W_{\Gamma}(X; \varphi_0^i)$ depends on M only through its first Betti number, which determines how many legs will be used to absorb the zero modes of χ , and therefore which curvature integrals must be performed on X . Combining eqs. (3.5), (3.25) and (3.40) we find that

$$Z_X(M) = |H_1(M, \mathbf{Z})|' \sum_{\Gamma \in \Gamma_{n, 3-b_1(M)}} b_{\Gamma}(X) \sum_a I_{\Gamma,a}, \quad (3.42)$$

where

$$b_{\Gamma}(X) = \frac{1}{(2\pi)^{2n}} \int_X W_{\Gamma}(X; \varphi_0^i) \sqrt{g} d^{4n} \varphi_0^i. \quad (3.43)$$

As we have already noted, when M is a rational homology sphere, the corresponding integrals $I_{\Gamma,a}$ are precisely the ones that have already appeared in the theory of perturbative Chern-Simons invariants. The “perturbative” sigma-model topological invariants appear as linear combinations of these integrals, albeit with weights, given in (3.43), that are seemingly different than in the Chern-Simons case. This is the basis for claim (I) in the introduction.

We asserted in the introduction that once one suspects that invariants of rational homology spheres can be constructed in this way, the invariance can be checked directly, without reference to physics. Let us see how this comes about. We assume some familiarity with the formal proof of metric-independence of Chern-Simons perturbation theory, as presented in [2], [3].

The basic idea is that when one varies with respect to the metric of M , one of the propagators changes by an exact form. After integration by parts, this causes another propagator to collapse to a delta function, giving a graph with one four-valent vertex in addition to the three-valent vertices. Each vertex has a factor of Ω_{IJKL} coupled to various fields. Upon collapsing a propagator, the Ω tensors at the end are joined by an ϵ^{IJ} – since the propagator is ϵ^{IJ} times a form on $M \times M$. The structure that one gets is thus $\epsilon^{I'J'} \Omega_{IJKL} \Omega_{I'J'K'L'}$ with the

free indices coupled to various fields. Most of the details do not matter, but one important point is that as each interaction vertex is linear in η , we have an η zero mode contracted with one index of each Ω , so the structure is more precisely $\epsilon^{II'}\Omega_{IJKL}\Omega_{I'J'K'L'}\eta^L\eta^{L'}$. The importance of this is that it means that we can assume antisymmetry in L and L' .

A given graph with one four-valent vertex and all other vertices trivalent can arise in three different ways by collapsing a line or propagator in a trivalent graph. Just as in Chern-Simons theory, the three contributions are identical except that the indices JK and $J'K'$ become permuted, so that the sum of the three contributions is proportional to

$$\epsilon^{II'} (\Omega_{IJKL}\Omega_{I'J'K'L'} + \Omega_{IJK'L}\Omega_{I'J'KL'} + \Omega_{IJJ'L}\Omega_{I'KK'L'}) - L \leftrightarrow L'. \quad (3.44)$$

The vanishing of the expression (3.44) as it appears inside the integral (3.43) amounts to the IHX relation between the weights of Feynman graphs. Therefore the following arguments constitute a rigorous proof of the IHX relation for the weights $b_\Gamma(X)$.

According to our previous identity (2.11), it is precisely the expression (3.44) that can be written as

$$\epsilon^{AB} (D_{AL}D_{BL'} - D_{AL'}D_{BL}) \Omega_{JKJ'K'}. \quad (3.45)$$

In Chern-Simons theory, instead of the quadratic function (3.44) of the curvature tensor, one obtains the quadratic function of the structure constants that vanishes according to the Jacobi identity. Here, rather than zero, we get a sort of total derivative, written in (3.45). But in contrast to Chern-Simons theory, there is an extra integral to do, namely the integral over X in eq. (3.43). The expression (3.45) does indeed disappear upon doing that integral. Suppose, for instance, that we take the first term in (3.45) and integrate the left-most derivative D_{AL} by parts. Resulting contributions would have D_{AL} acting on the Ω tensor at one or another vertex. This gives an expression $D_{AL}\Omega_{STUV}$ with indices contracted with various fields. Once again many details are irrelevant. The material point is that L is contracted with an η field – the antisymmetry in L and L' in (3.44) came because L and L' were contracted with η fields – and likewise (since all vertices in the minimal Lagrangian

are linear in η) one vertex in Ω_{STUV} , say V , is contracted with η . So we actually have something like $\eta^L \eta^V D_{AL} \Omega_{STUV}$, and this vanishes because of the Bianchi identity (2.5). (Of course, a contribution involving a derivative of η also vanishes because η is to be contracted with a covariantly constant zero mode.) This completes the direct formal argument for the IHX relation of the weight system (3.43) and for metric independence of the partition function (3.42).

At this point we would like to recall the following well-known fact (see, e.g. [11]): the IHX relation implies that the weight function vanishes on one-particle reducible graphs, i.e. on the connected graphs that can be made disconnected by removing one edge. Therefore, since we have just established the IHX relation for the weight system $b_\Gamma(X)$, the one-particle reducible diagrams can be excluded from the list of minimal Feynman diagrams that we have to consider in calculating the Feynman diagram contribution $Z_{\eta\chi\varphi}(M, X; \varphi_0^i)$.

By now we have enough information about the structure of Feynman diagrams in order to predict the behavior of the invariant $Z_X(M)$ under the change of orientation of M . Since the signs of the zero mode expectation values (3.21), (3.22) do not depend on the orientation of M , the only effect that its change has on $Z_X(M)$ comes from changing the sign in front of the tensor $\epsilon^{\mu\nu\rho}$ and hence in front of the Lagrangian L_2 in the action (2.13). As a result, the vertex V_1 and the propagator $\chi\chi$ change signs. Consider a minimal Feynman diagram containing m vertices V_1 and $2n - m$ vertices V_2 . The number of $\chi\chi$ propagators in this diagram is $n(1 - b_1) + m$. Therefore if the orientation of M is changed, then the invariant $Z_X(M)$ acquires a sign factor

$$(-1)^{n(1+b_1)}. \quad (3.46)$$

3.5 Feynman diagrams with operator insertions

Now let us see what happens when the operators (2.24) are included in the path integral. The role of the operators \mathcal{O}_η of equation (2.24) is very simple. Each of them absorbs l zero modes of η , and they do not interfere with Feynman diagrams. Therefore if we insert

operators \mathcal{O}_η with m fields η , then a minimal Feynman diagram should contain $2(n - m)$ vertices (3.28), (3.29) and $n - m + 1$ loops. In other words, the operators \mathcal{O}_η simply reduce the “effective” dimension of the hyper-Kähler manifold X .

In order to analyze the influence of the operators $\mathcal{O}_\alpha(\mathcal{K})$, we expand the path-ordered exponential into a sum of cyclically ordered integrals

$$\mathcal{O}_\alpha(\mathcal{K}) = \text{Tr}_\alpha \sum_{k=0}^{\infty} \frac{1}{k} \epsilon^{J_1 I_2} \dots \epsilon^{J_{k-1} I_k} \int_{\text{cycl. ord.}} A_{\mu_1 I_1 J_1}(x_1) \dots A_{\mu_k I_k J_k}(x_k) dx_1^{\mu_1} \dots dx_k^{\mu_k}. \quad (3.47)$$

Here Tr_α refers to an uncontracted pair of indices I_1, J_k . The analysis of diagrams with operators (3.47) is absolutely similar to that for the operators $V_{1,2}$. Indeed, the operators contained in the expansion of $A_{\mu I J}$ in fluctuating fields $\varphi^i, \chi_\mu^I, \eta^I$ are at least quadratic, and at the same time, they do not depend on \hbar . Therefore the minimal Feynman diagram in the presence of operators $\mathcal{O}_\alpha(\mathcal{K})$ contains only the “modifying” connection

$$B_{\mu I J} = \Omega_{I J K L} \chi_\mu^K \eta^L \quad (3.48)$$

and the more obvious pullback term can be dropped. We hence may use the simplified version of the expansion (3.47)

$$\mathcal{O}_\alpha(\mathcal{K}) = \text{Tr}_\alpha \sum_{k=0}^{\infty} \frac{1}{k} \epsilon^{J_1 I_2} \dots \epsilon^{J_{k-1} I_k} \int_{\text{cycl. ord.}} B_{\mu_1 I_1 J_1}(x_1) \dots B_{\mu_k I_k J_k}(x_k) dx_1^{\mu_1} \dots dx_k^{\mu_k}. \quad (3.49)$$

A minimal Feynman diagram contains altogether $2n$ vertices $V_1, V_2, B_{\mu I J}$. Therefore, if $\mathcal{L} \subset M$ is an N -component link with $Sp(n)$ representations \mathcal{V}_{α_b} assigned to its components \mathcal{L}_b , $1 \leq b \leq N$, then the partition function

$$Z_{\alpha_1, \dots, \alpha_N}(M, L, X) = \int \mathcal{D}\phi^i \mathcal{D}\eta^I \mathcal{D}\chi_\mu^I e^{-\frac{1}{\hbar} S} \prod_{b=1}^N \text{Tr}_{\alpha_b} \left(\text{Pexp} \oint_{\mathcal{L}_b} A_{\mu I J} dx^\mu \right) \quad (3.50)$$

can be expressed as a sum over Feynman graphs

$$Z_{\alpha_1, \dots, \alpha_N}(M, L, X) = |H_1(M, \mathbf{Z})|' \sum_{\Gamma \in \Gamma_{n, 3-b_1(M), N}} b_\Gamma(X) \sum_a I_{\Gamma, a}. \quad (3.51)$$

Here $\Gamma_{n, 3-b_1(M), N}$ is a set of $(3 - b_1(M))$ -valent graphs with legs attached to the components of the link \mathcal{L} , while $I_{\Gamma, a}$ are multiple integrals of the products of Green’s functions over M and

components of \mathcal{L} , which have already appeared in Chern-Simons perturbative calculations. The weights $b(\Gamma)$ are integrals (3.43). The integrands $W_\Gamma(X; \varphi_0^i)$ are calculated by the formula which is similar to eq. (3.41).

To be more specific, consider a knot \mathcal{K} in a rational homology sphere M . Let Γ be a graph with $2n - m$ trivalent vertices, $3n - 2m$ edges attached to trivalent vertices and m legs attached to the knot \mathcal{K} . A map σ from the set of pairs (k, l) , $1 \leq k \leq 3n - 2m$, $l = 1, 2$ into the set of pairs (i, j) , $1 \leq i \leq (2n - m)$, $j = 1, 2, 3$ still describes how the edges are attached to the vertices of Γ . We assign numbers $1, \dots, m$ to the legs in the order in which they are attached to \mathcal{K} . Then there is a map σ' from $1, \dots, m$ into a set of pairs (i, j) , $1 \leq i \leq 2n - m$, $j = 1, 2, 3$, which shows how legs are attached to trivalent vertices of Γ . Let s be a permutation which maps a set of pairs

$$(1, 0), (1, 1), (1, 2), (1, 3), \dots, (2n - m, 3), (1, 0)', (1, 1)', \dots, (m, 0)', (m, 1)'$$

onto a set

$$(1, 0), \dots, (2n - m, 0), (1, 0)', \dots, (m, 0)', \\ \sigma(1, 1), \sigma(1, 2), \dots, \sigma(3n - 2m, 1), \sigma(3n - 2m, 2), (1, 1)', \sigma'(1), \dots, (m, 1)', \sigma(m).$$

Then

$$W_\Gamma(X; \varphi_0^i) = (-1)^{|s|} \epsilon^{I_{1,0} \dots I_{2n-m,0} I'_{1,0} \dots I'_{m,0}} \prod_{k=1}^{3n-2m} \epsilon^{I_{\sigma(k,1)} I_{\sigma(k,2)}} \prod_{l=1}^m \epsilon^{I'_{l,1} I_{\sigma'(l)}} \prod_{j=1}^{m-1} \epsilon^{J'_{j,1} J'_{j+1,0}} \\ \times \text{Tr}_\alpha \prod_{p=1}^{2n-m} \Omega_{I_{p,0} I_{p,1} I_{p,2} I_{p,3}} \prod_{q=1}^m \Omega_{I'_{q,0} I'_{q,1} J'_{q,0} J'_{q,1}}. \quad (3.52)$$

Here Tr_α refers to the uncontracted pair of indices $J'_{1,0}, J'_{m,0}$.

3.6 Perturbative Chern-Simons theory

We will want to make a precise comparison of certain sigma-model invariants to Chern-Simons invariants. Therefore, we pause to recall some features of Chern-Simons perturbation theory.

We consider a Chern-Simons theory based on a simple Lie group G . Choose a basis x_a in the Lie algebra of G . We denote by $f_{ab}{}^c$ the structure constants

$$[x_a, x_b] = f_{ab}{}^c x_c. \quad (3.53)$$

Let h_{ab} be the Killing form such that the length of long roots is $\sqrt{2}$. Then the Chern-Simons action on a 3-dimensional manifold M can be written as

$$S_{\text{CS}} = -\frac{1}{2} \int_M \sqrt{h} d^3x \left(\frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} h_{ab} A_\mu^a \partial_\nu A_\rho^b + \frac{1}{3} \frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} f_{abc} A_\mu^a A_\nu^b A_\rho^c \right). \quad (3.54)$$

The quantum Chern-Simons invariant $Z_{\text{CS}}(M; k)$, $k \in \mathbf{Z}$ is expressed as a path integral

$$Z_{\text{CS}}(M; k) = \frac{1}{\text{Vol}(\tilde{G})} \int \mathcal{D}A_\mu^a \exp \left(\frac{i}{\hbar_{\text{CS}}} S_{\text{CS}} \right), \quad (3.55)$$

$$\hbar_{\text{CS}} = \frac{2\pi}{K}. \quad (3.56)$$

Here $\text{Vol}(\tilde{G})$ is the volume of the group of gauge transformations and $K = k + \nu$, ν being the dual Coxeter number of G . The choice of $\hbar_{\text{CS}} = \frac{2\pi}{K}$ rather than $\hbar_{\text{CS}} = \frac{2\pi}{k}$ is intended to account for a subtle one-loop effect that shifts the effective value of k seen in perturbation theory relative to what is seen with different methods of computation.

In the large K approximation, the path integral (3.55) can be presented as a sum of contributions coming from connected pieces of the moduli space of flat G connections over M . In what follows, we assume that M is a rational homology sphere, or, in other words, that

$$\dim H_1(M, \mathbf{R}) = 0. \quad (3.57)$$

The trivial connection on such a manifold is an isolated point in the moduli space of flat connections. We will use perturbation theory to evaluate the leading large K terms in the contribution of the trivial connection to the invariant (3.55):

$$Z_{\text{CS}}^{(\text{tr})}(M; k) = \frac{1}{\text{Vol}(\tilde{G})} \int_{[\text{triv}]} \mathcal{D}A_\mu^a \exp \left(\frac{i}{\hbar_{\text{CS}}} S_{\text{CS}} \right). \quad (3.58)$$

Here $\int_{[\text{triv}]}$ means that we take the trivial connection contribution to the path integral in the stationary phase approximation.

A perturbative calculation of the integral (3.58) requires gauge fixing. We choose the gauge

$$\partial^\mu A_\mu = 0. \quad (3.59)$$

We implement the gauge choice by introducing a scalar field t^a and a pair of fermionic ghost fields c^a, \bar{c}^a . After adding the gauge fixing terms, the action (3.54) becomes

$$\begin{aligned} S_{\text{CS,gf}} = \int_M \sqrt{h} d^3x \left(-\frac{1}{2} \frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} h_{ab} A_\mu^a \partial_\nu A_\rho^b - \frac{1}{6} \frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} f_{abc} A_\mu^a A_\nu^b A_\rho^c \right. \\ \left. + h_{ab} t^a \partial^\mu A_\mu^b + i h_{ab} \partial^\mu \bar{c}^a \partial_\mu c^b - i f_{abc} \bar{c}^a A_\mu^b \partial^\mu c^c \right). \end{aligned} \quad (3.60)$$

The propagators of the gauge and ghost fields are

$$\langle A_\mu^a(x_1) A_\nu^b(x_2) \rangle = -i \hbar_{\text{CS}} h^{ab} G_{\mu\nu}^{(\chi)}(x_1, x_2), \quad (3.61)$$

$$\langle c^a(x_1) \bar{c}^b(x_2) \rangle = -\hbar_{\text{CS}} h^{ab} G^{(\varphi)}(x_1, x_2). \quad (3.62)$$

The one-loop correction is known to be

$$Z_{\text{CS}}^{(\text{tr},0)}(M; k) = \frac{1}{\text{Vol}(G)} \left[\frac{2\pi \hbar_{\text{CS}}}{|H_1(M, \mathbf{Z})|^{\frac{1}{2}}} \right]^{\frac{1}{2} \dim G}; \quad (3.63)$$

here $\text{Vol}(G)$ is the volume of the gauge group G calculated with the Killing metric h_{ab} . The contribution of the trivial connection takes the form

$$Z_{\text{CS}}^{(\text{tr})}(M; k) = Z_{\text{CS}}^{(\text{tr},0)}(M; k) \exp \left[\sum_{n=1}^{\infty} \hbar_{\text{CS}}^n S_{n+1}(M) \right], \quad (3.64)$$

where $S_n(M)$ is the n -loop correction calculated according to the Feynman rules based on propagators (3.61), (3.62) and cubic interaction vertices

$$V_{1,\text{CS}} = -\frac{1}{6} \frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} f_{abc} A_\mu^a A_\nu^b A_\rho^c, \quad (3.65)$$

$$V_{2,\text{CS}} = -i f_{abc} \bar{c}^a A_\mu^b \partial^\mu c^c \quad (3.66)$$

coming from the gauge fixed action (3.60).

4 Atiyah-Hitchin manifold and Casson-Walker invariant of rational homology spheres

Our goal here is to make a precise comparison of certain invariants derived from topological sigma-models with the Casson-Walker invariant.

The reason that such a relation should exist was explained at the end of the introduction. The $N = 4$ supersymmetric gauge theory in three dimensions, with gauge group $SU(2)$, reduces at long distances [16] to a supersymmetric sigma model in which the target space is a certain non-compact hyper-Kähler manifold X_{AH} . The Casson-Walker invariant of a three-manifold M is (apart from a factor of two explained at the end of this section) the partition function of a topologically twisted form of the $N = 4$ gauge theory on M . It can be computed using any metric. If one scales up the metric h on M by $h \rightarrow th$ with $t \rightarrow \infty$, then one can replace the gauge theory by the sigma-model – and the topologically twisted gauge theory goes over to the topological sigma-model with target space X_{AH} . The precise X_{AH} that is relevant here is the moduli space of vacua of the $N = 4$ supersymmetric gauge theory. In fact, according to [16], X_{AH} is the “reduced moduli space” of a pair of BPS monopoles on \mathbf{R}^3 . It is an asymptotically flat hyper-Kähler manifold with fundamental group \mathbf{Z}_2 ; its geometry has been described in considerable detail in [17].

On the other hand, from what we have seen, for *any* four-dimensional hyper-Kähler manifold X , the partition function of the topological sigma-model will be a multiple of the two-loop Chern-Simons invariant. The Casson-Walker invariant will therefore be a multiple of the two-loop Chern-Simons invariant, as has been found before in a quite different way [8], [9].

We will first write down the sigma-model partition function $Z_X(M)$ with a general four-dimensional hyper-Kähler manifold. Then we will specialize to the case that X is the manifold X_{AH} , and extract the precise statement about the Casson-Walker invariant.

We will consider the calculation of $Z_X(M)$ for four types of manifolds M for which generally $Z_X(M) \neq 0$ - the manifolds with the first Betti number equal to 0 (i.e. rational

homology spheres), 1, 2 and 3.

4.1 Rational homology spheres

We start with the sigma-model. There is only one two-loop one-particle irreducible graph with two trivalent vertices. We call this graph θ . Both vertices in the Feynman diagrams of θ can be either V_1 or V_2 . The corresponding three-manifold integrals are

$$I_{\theta,1}(M) = \int_M \epsilon^{\mu_1\mu_2\mu_3} \epsilon^{\nu_1\nu_2\nu_3} G_{\mu_1\nu_1}^{(\chi)}(x_1, x_2) G_{\mu_2\nu_2}^{(\chi)}(x_1, x_2) G_{\mu_3\nu_3}^{(\chi)}(x_1, x_2) d^3x_1 d^3x_2, \quad (4.1)$$

$$I_{\theta,2}(M) = \int_M h^{\mu_1\mu_2}(x_1) h^{\nu_1\nu_2}(x_2) G_{\mu_1\nu_1}^{(\chi)}(x_1, x_2) \times \left(\frac{\partial}{\partial x_1^{\mu_2}} G^{(\varphi)}(x_1, x_2) \right) \left(\frac{\partial}{\partial x_2^{\nu_2}} G^{(\varphi)}(x_1, x_2) \right) \sqrt{h(x_1)} d^3x_1 \sqrt{h(x_2)} d^3x_2. \quad (4.2)$$

The weight function $b_\theta(X)$ is equal to the integral

$$b_\theta(X) = \frac{1}{4\pi^2} \int_X \sqrt{g} d^4\varphi_0^i \epsilon^{I_1J_1} \epsilon^{I_2J_2} \epsilon^{I_3J_3} \epsilon^{I_4J_4} \Omega_{I_1I_2I_3I_4} \Omega_{J_1J_2J_3J_4} = \frac{1}{8\pi^2} \text{Tr} \int_X R \wedge R. \quad (4.3)$$

Here R is the curvature of X . The latter integral was calculated [24] for the case when X is the Atiyah-Hitchin manifold X_{AH} :

$$b_\theta(X_{\text{AH}}) = \frac{1}{8\pi^2} \text{Tr} \int_{X_{\text{AH}}} R \wedge R = -2. \quad (4.4)$$

The authors of [24] integrated over the simply-connected double cover of X_{AH} , so we multiplied their result by 1/2.

Now let us calculate the combinatorial factors. Consider first the diagram with 2 vertices V_1 . Each vertex carries a factor of -1 (-1 comes from $e^{-\frac{1}{\hbar}S}$ in eq. (3.3), $\frac{1}{6}$ comes from eq. (3.3) and $3!$ comes from the fact that V_1 has 3 identical legs). The diagram is symmetric under permutations of 2 vertices and 3 propagators, which produces a factor of $\frac{1}{12}$. Rearranging the fermionic fields requires an even number of permutations, so no sign factor comes from there. Thus we find that the diagram with two vertices V_1 contributes

$$\frac{1}{12} |H_1(M, \mathbf{Z})| I_{\theta,1}(M) b_\theta(X) \quad (4.5)$$

Consider now the diagram with two vertices V_2 . Each vertex carries a factor of -1 from e^{-S} . The diagram has a symmetry of permuting the vertices, hence the factor $\frac{1}{2}$. Rearranging the fermionic fields produces -1 . Each propagator $\langle \varphi \overline{\varphi} \rangle$ carries -1 . An extra factor -1 comes from arranging the indices of tensors Ω into the contraction of eq. (4.3). Therefore the diagram with two vertices V_2 contributes

$$\frac{1}{2} |H_1(M, \mathbf{Z})| I_{\theta,2}(M) b_\theta(X). \quad (4.6)$$

As a result,

$$Z_X(M) = \frac{1}{2} b_\theta(X) |H_1(M, \mathbf{Z})| \left(\frac{1}{6} I_{\theta,1}(M) + I_{\theta,2}(M) \right). \quad (4.7)$$

A calculation of the two-loop correction $S_2(M)$ to the trivial connection contribution to the quantum Chern-Simons invariant goes along the similar lines. First, we consider a diagram with two vertices $V_{1,\text{CS}}$. Its contribution is proportional to $I_{\theta,1}(M)$. The coefficient is composed of the following factors: $-i$ from every vertex (3.65), $-i$ from every propagator (3.61), $\frac{1}{12}$ from the symmetry of the diagram and

$$f^{abc} f_{abc} = 2\nu \dim G \quad (4.8)$$

from contracting the Lie algebra indices. Here ν is the dual Coxeter number of G . Thus the contribution to $S_2(M)$ is

$$-\frac{i}{6} \nu \dim G I_{\theta,1}(M). \quad (4.9)$$

Now we turn to the diagram with 2 vertices $V_{2,\text{CS}}$. Apart from $I_2(M)$, we find the following factors: 1 from every vertex, $-i$ from the propagator (3.61), -1 from permuting the fermions, $\frac{1}{2}$ from the symmetry of the diagram and the factor (4.8) with extra -1 from contracting the Lie algebra indices. Thus the contribution of the diagram with 2 vertices $V_{2,\text{CS}}$ is

$$-i\nu \dim G I_{\theta,2}(M) \quad (4.10)$$

and, as a result,

$$S_2(M) = -i\nu \dim G \left(\frac{1}{6} I_{\theta,1}(M) + I_{\theta,2}(M) \right). \quad (4.11)$$

Comparing eqs. (4.7) and (4.11) we find that

$$Z_X(M) = \frac{i}{2} b_\theta(X) |H_1(M, \mathbf{Z})| \frac{S_2(M)}{\nu \dim G} \quad (4.12)$$

and, in particular,

$$Z_{X_{\text{AH}}}(M) = -i |H_1(M, \mathbf{Z})| \frac{S_2(M)}{\nu \dim G}. \quad (4.13)$$

It should be noted that the sum of integrals

$$\frac{1}{6} I_{\theta,1}(M) + I_{\theta,2}(M) \quad (4.14)$$

appearing in eqs. (4.7) and (4.11) depends on the metric $h_{\mu\nu}$ of the manifold M . This dependence can be compensated [25] by adding a special counterterm $\frac{1}{48\pi} I(h, \text{fr})$. Suppose that M is equipped with a local basis in its tangent bundle. The topological class of the basis defines a framing fr of M . Let ω be a Levi-Civita connection of M relative to this basis. Then

$$I(h, \text{fr}) = \frac{1}{4\pi} \int_M \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right). \quad (4.15)$$

If the framing is changed by 1 unit then the integral (4.15) changes by 2π . Since we expect both $Z_X(M)$ and $S_2(M)$ to be topological invariants, we assume that the counterterm $\frac{1}{48\pi} I(h, \text{fr})$ is added to the sum (4.14):

$$Z_X(M, \text{fr}) = -\frac{1}{2} b_\theta(X) |H_1(M, \mathbf{Z})| \left(\frac{1}{6} I_{\theta,1}(M) + I_{\theta,2}(M) + \frac{1}{48\pi} I(h, \text{fr}) \right), \quad (4.16)$$

$$S_2(M, \text{fr}) = i\nu \dim G \left(\frac{1}{6} I_{\theta,1}(M) + I_{\theta,2}(M) + \frac{1}{48\pi} I(h, \text{fr}) \right). \quad (4.17)$$

We paid a price for making $Z_X(M)$ and $S_2(M)$ topological invariants: they now depend on the choice of framing. This dependence is well-known for $S_2(M)$. Fortunately, for any three-manifold M there is a preferred *canonical* framing fr_0 . Therefore $S_2(M, \text{fr})$ and $Z_X(M, \text{fr})$

can be converted into true topological invariants of M if they are calculated at $\text{fr} = \text{fr}_0$. A reduction to the canonical framing is achieved with the help of relations

$$Z_X(M, \text{fr}) = Z_X(M) - \frac{1}{48} b_\theta(X) |H_1(M, \mathbf{Z})| \Delta \text{fr}, \quad (4.18)$$

$$S_2(M, \text{fr}) = S_2(M) + \frac{1}{24} i\nu \dim G \Delta \text{fr}. \quad (4.19)$$

Here $Z_X(M)$ and $S_2(M)$ denote the invariants calculated at the canonical framing, and Δfr is the framing correction:

$$\Delta \text{fr} = \text{fr} - \text{fr}_0. \quad (4.20)$$

There is actually an important point that should be made here about the topological sigma-model. If this is regarded as a theory in its own right, it requires some choice of framing, and there is nothing more to say. But in some cases – such as the case that the target is the Atiyah-Hitchin manifold X_{AH} – the sigma-model can be viewed as the low energy limit of a model (in this case a supersymmetric gauge theory) with much softer ultraviolet behavior. When this is so, as the soft theory will not generate a two-loop framing anomaly, the sigma-model will be endowed with a natural framing if it is viewed as the low energy limit of the gauge theory. It is natural to suspect that this framing is $\text{fr} = \text{fr}_0$, and we will assume this in writing the following formulas.

Let λ_{CW} be the Casson-Walker invariant of M . We will use it also in a slightly different normalization:

$$\lambda_{\text{C}} = |H_1(M, \mathbf{Z})| \lambda_{\text{CW}}. \quad (4.21)$$

The normalization of λ_{C} seems to be more in line with Casson's original definition: apart from the contribution of reducible flat connections $SU(2)$, it counts the Euler characteristics of the moduli spaces of irreducible flat $SU(2)$ connections. It is known [8], [9] that $S_2(M, \text{fr})$ calculated at canonical framing is proportional to λ_{CW} :

$$S_2(M, \text{fr}_0) = \frac{i}{2} \nu \dim G \lambda_{\text{CW}}(M). \quad (4.22)$$

Therefore eq. (4.13) is compatible with (1.7) or, more generally, with

$$Z_{X_{\text{AH}}}(M, \text{fr}) = \frac{1}{2}(-1)^{b_1} \lambda_C + \frac{1}{24} |H_1(M, \mathbf{Z})| \Delta \text{fr}, \quad (4.23)$$

where Δfr is the framing correction (4.20).

The factor of $1/2$ in (1.7) has the following natural interpretation. In topological $SU(2)$ gauge theory (which comes from the twisting of the $N = 4$ supersymmetric 3-dimensional theory), a gauge transformation by the constant element -1 of the center of $SU(2)$ acts trivially on the space of connections. Therefore, an isolated, irreducible $SU(2)$ connection is invariant under a group of order two, consisting of gauge transformations by ± 1 , and contributes $\pm 1/2$, rather than ± 1 , to the partition function of the topological field theory. This factor of $1/2$ is omitted in the usual definition of the Casson-Walker invariant (although originally Casson put it in motivated by the fact that the invariant λ_C appeared to be always even for integer homology spheres).

4.2 Manifolds with $b_1(M) = 1$

Let M be a manifold with $b_1(M) = 1$. Let $\omega_\mu^{(1)}$ be a one-form on M which represents the integral cohomology class. The only minimal Feynman graph contributing to $Z_X(M)$ is the loop with two vertices sitting on it. Both vertices can be either V_1 or V_2 . The corresponding integrals of Green's functions are

$$I_1(M) = \int_M \epsilon^{\mu_1 \mu_2 \mu_3} \epsilon^{\nu_1 \nu_2 \nu_3} \omega_{\mu_1}^{(1)}(x_1) \omega_{\nu_1}^{(1)}(x_2) G_{\mu_2 \nu_2}^{(\chi)}(x_1, x_2) G_{\mu_3 \nu_3}^{(\chi)}(x_1, x_2) d^3 x_1 d^3 x_2, \quad (4.24)$$

$$I_2(M) = \int_M h^{\mu_1 \mu_2}(x_1) h^{\nu_1 \nu_2}(x_2) \omega_{\mu_1}^{(1)}(x_1) \omega_{\nu_1}^{(1)}(x_2) \times \left(\frac{\partial}{\partial x_1^{\mu_2}} G^{(\varphi)}(x_1, x_2) \right) \left(\frac{\partial}{\partial x_2^{\nu_2}} G^{(\varphi)}(x_1, x_2) \right) \sqrt{h(x_1)} d^3 x_1 \sqrt{h(x_2)} d^3 x_2. \quad (4.25)$$

The accompanying weight function is still that of eq. (4.3). The combinatorial factor for I_2 is the same as for the case of $b_1(M) = 0$. The combinatorial factor for I_1 is 3 times bigger, because the symmetry permutes only two propagators. As a result,

$$Z_X(M) = \frac{1}{2} b_\theta(X) |H_1(M, \mathbf{Z})|' \left(\frac{1}{2} I_1(M) + I_2(M) \right). \quad (4.26)$$

The same combination of integrals as in eq. (4.26) appears in Chern-Simons theory on M . To be specific, consider a Chern-Simons theory with the gauge group $SU(2)$. The basis of the Lie algebra $su(2)$ is formed by Pauli σ -matrices: $i\sigma_1, i\sigma_2, i\sigma_3$. The corresponding structure constants are

$$f_{ab}{}^c = -2\epsilon_{abc}. \quad (4.27)$$

Here ϵ_{abc} is an antisymmetric tensor with $\epsilon_{123} = 1$.

For an arbitrary real parameter u , consider a flat connection on M

$$A_\mu^a = \delta_3^a u \omega_\mu^{(1)}. \quad (4.28)$$

The one-loop contribution of the fields

$$A_\mu^{1,2}, c^{1,2}, \bar{c}^{1,2}, t^{1,2} \quad (4.29)$$

to the Chern-Simons partition function at the background (4.28) is known to be equal to the inverse Reidemeister torsion $[\tau_R(M; e^{2iu})]^{-1}$. The torsion has the following expansion at small u :

$$\tau_R(M; e^{2iu}) = \frac{1}{4u^2} |H_1(M, \mathbf{Z})|' \left(1 + \sum_{m=1}^{\infty} C_m(M) u^{2m} \right). \quad (4.30)$$

Here $C_n(M)$ are some invariants of M . The prefactor $1/4u^2$ is due to the zero modes. The coefficient $-C_1(M)$ is the quadratic contribution to the inverse Reidemeister torsion. It is calculated by the same diagrams that led to eq. (4.26), except that we have to use either two vertices (3.65) or two vertices (3.66). The calculation of combinatorial factors is similar to that for eq. (4.11) with a few exceptions. $I_1(M)$ acquires an extra factor of 3 due to the reduction of symmetry between the propagators, there is an overall extra factor of i/\hbar_{CS} because one pairing (3.61) is substituted by two fields (4.28), and we should put $f_{3ab}f_3^{ab} = 8$ instead of $2\nu \dim G$. As a result,

$$-C_1(M) = 4 \left(\frac{1}{2} I_1(M) + I_2(M) \right). \quad (4.31)$$

Comparing this with eq. (4.26) we conclude that

$$Z_X(M) = -\frac{1}{4} b_\theta(X) \left(u^2 \tau_R(M; e^{iu}) \right)''_{u=0} \quad (4.32)$$

and, in particular,

$$Z_{X_{\text{AH}}}(M) = \frac{1}{2} \left(u^2 \tau_{\text{R}}(M; e^{iu}) \right)''_{u=0}. \quad (4.33)$$

This formula is in agreement with eq. (1.7) and the results of [20].

If $M = S^2 \times S^1$, then $\tau_{\text{R}}(M; e^{iu}) = \frac{1}{4 \sin^2(\frac{u}{2})}$ and

$$Z_X(S^2 \times S^1) = -\frac{1}{24} b_{\theta}(X), \quad Z_{X_{\text{AH}}}(S^2 \times S^1) = \frac{1}{12}. \quad (4.34)$$

Another useful example of an application of eq. (4.33) is a torus bundle over a circle. This is a 3-manifold constructed by gluing two T^2 boundaries of a 3-manifold $T^2 \times [0, 1]$ through an $SL(2, \mathbf{Z})$ twist

$$U = \begin{pmatrix} p & r \\ q & s \end{pmatrix}, \quad ps - qr = 1. \quad (4.35)$$

The Reidemeister torsion of the manifold T_U constructed in this way is

$$\tau_{\text{R}}(T_U; e^{iu}) = \frac{e^{-iu}}{4 \sin^2(\frac{u}{2})} \det(U - e^{iu} I) = \frac{2 \cos u - (p + s)}{4 \sin^2(\frac{u}{2})}. \quad (4.36)$$

As a result,

$$Z_X(T_U) = \frac{1}{24} b_{\theta}(X) (p + s + 10). \quad (4.37)$$

4.3 Manifolds with $b_1(M) = 2$

Let M be a manifold with $b_1(M) = 2$. Let $\omega_{\mu}^{(\alpha)}$, $\alpha = 1, 2$ be the basic integral one-forms. The only minimal Feynman diagram contributing to $Z_X(M)$ consists of two vertices V_1 connected by a single propagator (3.38). All remaining legs absorb the zero modes of η^I and χ^I . The corresponding integral is

$$I_1(M) = \int_M \epsilon^{\mu_1 \mu_2 \mu_3} \epsilon^{\nu_1 \nu_2 \nu_3} \omega_{\mu_1}^{(1)}(x_1) \omega_{\nu_1}^{(1)}(x_2) \omega_{\mu_2}^{(2)}(x_1) \omega_{\nu_2}^{(2)}(x_2) G_{\mu_3 \nu_3}^{(\chi)}(x_1, x_2) d^3 x_1 d^3 x_2, \quad (4.38)$$

This can be evaluated in the following way. Let c be the two-form $c = \omega^{(1)} \wedge \omega^{(2)}$. Then as $\omega^{(1)} \wedge c = \omega^{(2)} \wedge c = 0$, it follows from Poincaré duality that the cohomology class of c is

trivial, so that $c = dg$, with g a one-form. Using the fact that the propagator G in (4.38) obeys $dG = \delta$ (where δ is a delta function supported on the diagonal in $M \times M$), it follows that $I_1 = \int_M g \wedge \omega^{(1)} \wedge \omega^{(2)}$, showing the appearance of the Massey product or Milnor linking number, as promised above.

The weight function is still $b_\theta(X)$ and the combinatorial factor is the one in eq. (4.26) times 2 due to the reduction of symmetry between the propagators. As a result,

$$Z_X(M) = \frac{1}{2} b_\theta(X) |H_1(M, \mathbf{Z})|' I_1(M), \quad (4.39)$$

and, in particular,

$$Z_{X_{\text{AH}}}(M) = -|H_1(M, \mathbf{Z})|' I_1(M). \quad (4.40)$$

The latter result is consistent with the calculations of [20]. The only difference is that the integral $I_1(M)$ was presented there in a ‘‘Poincaré dual’’ way: as a self-linking number of the intersection of two 2-cycles which are dual to the forms $\omega_\mu^{(\alpha)}$.

4.4 Manifolds with $b_1(M) = 3$

Let M be a manifold with $b_1(M) = 3$. Let $\omega_\mu^{(\alpha)}$, $\alpha = 1, 2, 3$ be one-forms giving a basis of the image of $H^1(M, \mathbf{Z})$ in $H^1(M, \mathbf{R})$. The only minimal Feynman diagram contributing to $Z_X(M)$ consists of two disconnected vertices V_1 , all their legs being attached to the zero modes. The corresponding integral is the square of the intersection number of the forms $\omega_\mu^{(\alpha)}$:

$$I_1 = \left(\int_M \epsilon^{\mu_1 \mu_2 \mu_3} \omega_{\mu_1}^{(1)}(x) \omega_{\mu_2}^{(2)}(x) \omega_{\mu_3}^{(3)}(x) d^3 x \right)^2. \quad (4.41)$$

This integral is one of the most obvious classical invariants of a three-manifold with $b_1(M) = 3$. The weight function is again $b_\theta(X)$. The combinatorial factor is the same that appeared at $I_1(M)$ in eq. (4.39) except that we use the zero-mode expectation value (3.22) instead of the remaining propagator (3.38). As a result,

$$Z_X(M) = \frac{1}{2} |H_1(M, \mathbf{Z})|' b_\theta(X) I_1(M), \quad Z_{X_{\text{AH}}}(M) = -|H_1(M, \mathbf{Z})|' I_1(M). \quad (4.42)$$

The latter equation is consistent with eq. (1.7) and with the calculations of [20].

Let M be a 3-dimensional torus T^3 . Since $|H_1(T^3, \mathbf{Z})|' = 1$ and $I_1(T^3) = 1$, we find that

$$Z_X(T^3) = \frac{1}{2}b_\theta(X), \quad Z_{X_{\text{AH}}}(T^3) = -1. \quad (4.43)$$

5 Hilbert spaces, operators and gluing properties of the Casson-Walker invariant

In this section, we will begin to analyze the “physical Hilbert spaces” obtained by quantizing the topological sigma-model on a Riemann surface Σ . This will give us information about the gluing properties of the Casson-Walker invariant in view of its relation to the partition function (3.3) of the topological sigma-model. Our analysis will be more complete for the case that Σ has genus zero or one.

Let M be a three-manifold with a boundary $\partial M = \Sigma$; here Σ is a Riemann surface. The path integral (3.3) for such an M has to be taken over the fields $\phi^i, \eta^I, \chi_\mu^I$ which satisfy certain boundary conditions on Σ . As a result, $Z_X(M)$ becomes a function of the boundary values. We denote this function as $|M\rangle$. All permissible functions of the boundary values on Σ form a Hilbert space⁶ \mathcal{H}_Σ^0 . In topological quantum field theories, one can try to restrict oneself to Q -invariant boundary conditions modulo the action of Q . (The present theory has two Q ’s and one can, for instance, simply use any linear combination of them.) Equivalently, one can construct the full Hilbert space and then identify the cohomology of Q as the physical Hilbert space \mathcal{H}_Σ appropriate to the topological field theory. The \mathcal{H}_Σ are often finite-dimensional, though the underlying Hilbert space \mathcal{H}_Σ^0 of all states, not necessarily Q -invariant, is always infinite-dimensional.

Suppose that the boundaries of two oriented 3-manifolds M_1 and M_2 are isomorphic to the same Riemann surface Σ . We can glue M_1 and M_2 together along Σ , thus obtaining a

⁶More properly, in a non-unitary theory such as the twisted theory considered here, these spaces are vector spaces with a non-degenerate inner product which is not positive. It is conventional in physics to call them Hilbert spaces anyway.

new manifold M . According to quantum field theory, the partition function of M can be expressed through a scalar product in the Hilbert space \mathcal{H}_Σ

$$Z_X(M) = \langle M_2^* | M_1 \rangle = \langle M_1^* | M_2 \rangle. \quad (5.1)$$

Here M^* denotes a manifold M with the opposite orientation.

We could twist a boundary of a manifold M_1 by an element U of the mapping class group of Σ prior to gluing. The mapping class group (or its central extension) is represented in \mathcal{H}_Σ , and this representation determines the partition function of the manifold M^U constructed by gluing M_1 and M_2 after a twist U

$$Z_X(M^U) = \langle M_2^* | U | M_1 \rangle. \quad (5.2)$$

The formulas (5.1) and (5.2) require some caution due to the problems related to the sign of the determinant (3.14). We will see in the next subsections that the path integral calculation of the states $|M_1\rangle$, $|M_2\rangle$ most easily determines them only up to a sign. A complete definition of these states requires a choice of orientation on the spaces of the zero modes of the Reidemeister torsion (with appropriate boundary conditions). For a manifold with boundary this choice is not canonical. A gluing of M_1 and M_2 equipped with orientations on the spaces of zero modes, induces an orientation on the space (3.19). This induced orientation may differ from the canonical one. In this case one has to put an extra negative sign in the left hand side of eqs. (5.1) and (5.2).

As we will see shortly, the space $\mathcal{H}_\Sigma(X)$ for $\Sigma = S^2, T^2$ is related to $\bar{\partial}$ -cohomology of certain classes of forms on X . If X is non-compact, for instance, the Atiyah-Hitchin manifold X_{AH} , then there are several kinds of $\bar{\partial}$ cohomology one might consider (e.g. ordinary cohomology, cohomology with compact support, and \mathbf{L}^2 cohomology) which give quite different answers. In general, the analysis by cutting and summing over physical states is likely to be quite subtle if X is not compact, roughly because there is a continuum of almost Q -invariant states starting at zero energy. In the presence of such a continuum, formal arguments claiming to show a reduction to the Q -cohomology are hazardous at best. But if X is compact,

the spectrum is discrete, and one will get a quite straightforward formalism involving a sum over finitely many physical states.

On the other hand, we know from subsection 3.4 that the dependence on X is very simple – X only enters via the values of certain curvature integrals. Our approach, therefore, will be to apply the cutting and pasting formalism for compact X , and infer the behavior for general X from the values of the curvature integrals.

For illustrative purposes and application to the Casson-Walker invariant, we will consider the case that X is four-dimensional. There is then only one relevant curvature integral (related to the fact that there is only one one-particle irreducible trivalent graph with two loops), so any four-dimensional compact hyper-Kähler manifold X with a non-zero value of this one integral can serve as a universal example. There is only one candidate: a K3 surface. Then if we want to make a statement, for example, about the Casson invariant, we must take X to be the non-compact manifold X_{AH} ; in going from K3 to X_{AH} we simply multiply the three-manifold invariant by a constant, which is the ratio of the appropriate weight functions. Since

$$b_\theta(\text{K3}) = 48, \tag{5.3}$$

we have for any 3-manifold M ,

$$Z_{\text{K3}}(M) = \frac{b_\theta(\text{K3})}{b_\theta(X_{\text{AH}})} Z_{X_{\text{AH}}}(M) = -\frac{1}{4} b_\theta(\text{K3}) \lambda_{\text{C}}(M) = -12 \lambda_{\text{C}}(M). \tag{5.4}$$

In order to find the structure of \mathcal{H}_Σ , we consider a manifold $M = \mathbf{R}^1 \times \Sigma$ with coordinates x^0 and (x^1, x^2) referring to \mathbf{R}^1 and Σ . The topological nature of the sigma-model means that we can try to construct the physical Hilbert space by quantizing the theory in the small \hbar limit. In doing so, the first step is to find the solutions of the linearized equations of motion obtained in expanding around a constant map from M to X , and then quantize. Using the invariance of M under time translations, one can expand the space of classical solutions in a basis of modes of definite frequency; by “zero modes” and “non-zero modes” we mean eigenmodes of zero or non-zero frequency. In the leading approximation, one gets a Fock

space \mathcal{F} of the non-zero modes of the various fields tensored with (or rather fibered over) a more complicated structure built from the zero modes. As is usual in such theories, Q is acyclic in \mathcal{F} away from the ground state, so we can throw away the non-zero modes and simply quantize the zero modes.⁷

The x^0 -independent classical fields for the action (3.7) include the constant bosonic fields ϕ^i , constant fermionic fields η^I, χ_0^I as well as the fields $\chi_\mu^I(x) = \chi_\alpha^I \omega_\mu^{(\alpha)}$, $\mu = 1, 2$, $1 \leq \alpha \leq \dim H^1(\Sigma)$, here $\omega^{(\alpha)}$ are harmonic one-forms on Σ , and χ_α^I are constant fermionic coefficients.

The first order structure of the fermionic part of the action (3.7) indicates that the fermionic fields satisfy the following anti-commutation relations (we assume here that $\hbar = 1$):

$$\{\eta^I, \chi_0^J\} = \epsilon^{IJ}, \quad (5.5)$$

$$\{\chi_\alpha^I, \chi_\beta^J\} = -\epsilon^{IJ} (L^{-1})_{\alpha\beta}. \quad (5.6)$$

Here $L^{\alpha\beta}$ is the matrix of intersection pairing on Σ :

$$L^{\alpha\beta} = \int_\Sigma \omega^{(\alpha)} \wedge \omega^{(\beta)}. \quad (5.7)$$

The formulas (5.5), (5.6) together with some additional considerations below lead us to propose the following structure of the Hilbert space \mathcal{H}_{Σ_g} for a genus g Riemann surface Σ_g and a compact hyper-Kähler manifold X . We recall that V is the natural $Sp(n)$ bundle over X . Let $\wedge^* V$ denote the sum of all exterior powers of V . Pick one of the complex structures on X . Then we conjecture that \mathcal{H}_{Σ_g} is the sum of $\bar{\partial}$ -cohomology groups of X with values in the g -th tensor power of the bundle $\wedge^* V$:

$$\mathcal{H}_{\Sigma_g} = \sum_{q=0}^{\dim_{\mathbf{C}} X} H_{\bar{\partial}}^q \left(X, (\wedge^* V)^{\otimes g} \right). \quad (5.8)$$

The basis for this formula will become clear as we analyze the cases $g = 0, 1$. To use the \mathcal{H}_{Σ_g} in relation to three-manifolds requires, however, more than the “additive” formula just

⁷An important source of simplification in the particular models considered here is that, as one varies the constant map to X , there are no “singularities” at which the separation between zero and non-zero modes breaks down. In other examples such as topological gauge theories related to Donaldson theory, such singularities are a prime cause of difficulty.

proposed. One also needs to understand the action of the mapping class group. We will analyze this action in detail for $g = 1$ but believe that there may be some subtlety for $g > 1$.

The Hilbert space \mathcal{H}_{Σ_g} is \mathbf{Z}_2 -graded. In other words, it splits into a sum of bosonic and fermionic subspaces, which are $+1$ and -1 eigenspaces of the fermionic parity operator $(-1)^F$. The relative grading of the states inside a given space \mathcal{H}_{Σ_g} is determined by the action of the fermionic operators η and χ . The absolute grading can be deduced from the consistency between the Feynman diagram and “Hilbert space super-trace” calculations of partition functions of some 3-manifolds. This consistency dictates that the subspace $H_{\partial}^0(X, (\Lambda^p V)^{\otimes g})$ has the fermionic parity $(-1)^{1+g}$. Then the fermionic parity of a subspace $H_{\partial}^q(X, \bigotimes_{i=1}^g \Lambda^{l_i} V)$ is $(-1)^{1+q+g+\sum_{i=1}^g l_i}$. We will see in the next subsections how the consistency check works in the case of $g = 0, 1$.

5.1 The Hilbert space of a 2-sphere and the formula for a connected sum of 3-manifolds

Let us consider first the case of $\Sigma = S^2$. Since $\dim H^1(S^2) = 0$, we have to deal only with the fermionic fields η^I and χ_0^I . The commutation relation (5.5) is represented in a 2^{2n} -dimensional space. This space contains the vacuum state $|0\rangle_{\eta}$ which is annihilated by χ_0^I :

$$\chi_0^I |0\rangle_{\eta} = 0, \quad (5.9)$$

and the states produced by the action of operators η^I :

$$\eta^{I_1} \cdots \eta^{I_l} |0\rangle_{\eta}. \quad (5.10)$$

Including also the bosons, which parametrize the choice of a point in X , the Hilbert space obtained by quantizing the zero modes is simply the space of sections of the exterior algebra $\Lambda^* V$ of the bundle $V \rightarrow X$. Indeed, the state

$$|\psi\rangle = \psi_{I_1 \dots I_l}(\phi) \eta^{I_1} \cdots \eta^{I_l} |0\rangle_{\eta} \quad (5.11)$$

is interpreted as a section of $\Lambda^l V$. The inner product between quantum field theory states becomes in this subspace the scalar product between sections of $\Lambda^l V$ and sections of $\Lambda^{2n-l} V$ defined as

$$\langle \psi^{(1)} | \psi^{(2)} \rangle = \frac{1}{(2\pi)^{2n}} \int_X \sqrt{g} d^{4n} \varphi_0^i \epsilon^{I_1 \dots I_{2n}} \psi_{I_1 \dots I_l}^{(1)} \psi_{I_{l+1} \dots I_{2n}}^{(2)}. \quad (5.12)$$

Note that we changed the order of indices in $\psi^{(1)}$.

The action of the operators Q_A can be inferred from the transformation laws $\delta \phi^i = \gamma_{AI}^i \epsilon^A \eta^I$, $\delta \eta^I = 0$. This shows that Q_A will act by differentiating ϕ and adding an η , thus mapping a section of $\Lambda^l V$ to a section of $\Lambda^{l+1} V$. The action of Q_A is in fact

$$Q_A \psi_{I_1 \dots I_l} \eta^{I_1} \dots \eta^{I_l} |0\rangle_\eta = \gamma_{I_0 A}^i D_i \psi_{I_1 \dots I_l} \eta^{I_0} \eta^{I_1} \dots \eta^{I_l} |0\rangle_\eta. \quad (5.13)$$

If out of the hyper-Kähler structure on X , we pick a particular complex structure, then $\Lambda^l V$ can be identified with the space of $(0, l)$ -forms on X , and one of the Q 's becomes the $\bar{\partial}$ operator. If X is compact, the space of physical states is thus the finite-dimensional space

$$\mathcal{H}_{S^2} = \bigoplus_{l=0}^{2n} H^{0,l}(X)$$

(cf. (5.8) for $g = 0$) and the fermionic parity of a subspace $H^{0,l}(X)$ is $(-1)^{1+l}$.

In the case of compact X , we can immediately determine the value of the partition function for the three-manifold $S^2 \times S^1$. The same arguments as in [1] indicate that it is the \mathbb{Z}_2 -graded (that is, super-) dimension of the physical Hilbert space, or

$$Z_X(S^2 \times S^1) = \text{sdim } \mathcal{H}_{S^2} = \sum_{l=0}^{2n} (-1)^{1+l} \dim H^{0,l}. \quad (5.14)$$

For instance, if X is of dimension four, then according to the index theorem for the $\bar{\partial}$ operator, the right hand side is

$$Z_X(S^2 \times S^1) = -\frac{1}{192\pi^2} \int_X \text{Tr}(R \wedge R) = -\frac{1}{24} b_\theta(X), \quad (5.15)$$

which is the same as eq. (4.34). For reasons explained above, eq. (5.15) is valid also for X non-compact but asymptotically flat. Notice that, in general, the right hand side of (5.15)

is not integral; for instance, if X is the Atiyah-Hitchin manifold X_{AH} , then using the value of the curvature integral for X_{AH} from eq. (4.4), we get $Z_{X_{\text{AH}}}(S^2 \times S^1) = -1/12$. This shows that the non-compactness and continuous spectrum are really essential; there is a real obstruction to representing the system, for manifolds such as X_{AH} , by a finite-dimensional space of physical states, which would necessarily give an integer for the partition function on $S^2 \times S^1$.

So to proceed, we take X to be a K3 manifold. The $(0, q)$ part of the $\bar{\partial}$ cohomology is very simple. $H^{0,0}$ is one-dimensional, represented by a constant function or equivalently the Fock ground state

$$\psi^{(0)} = |0\rangle_\eta. \quad (5.16)$$

Also $H^{0,1} = 0$, and $H^{0,2}$ is one-dimensional, represented by a state that we can think of as

$$\psi^{(2)} = \mathcal{O}_\eta(\omega)\psi^{(0)}, \quad (5.17)$$

where $\mathcal{O}_\eta(\omega)$ is the BRST-invariant operator (2.24), and we choose a $(2, 0)$ -form ω to be

$$\omega_{I_1 I_2} = \begin{cases} -\frac{2\pi^2}{\text{Vol}(X)} \epsilon_{I_1 I_2} & \text{for } X = \text{K3 (} X\text{-compact)}, \\ \frac{1}{b_\theta(X)} \epsilon^{J_1 J_2} \epsilon^{K_1 K_2} \epsilon^{L_1 L_2} \Omega_{I_1 J_1 K_1 L_1} \Omega_{I_2 J_2 K_2 L_2} & \text{for } X = X_{\text{AH}} (X\text{-non-compact)}. \end{cases} \quad (5.18)$$

According to eq. (5.12), with this normalization of the form ω , the scalar product in \mathcal{H}_{S^2} is

$$\langle \psi^{(0)} | \psi^{(0)} \rangle = \langle \psi^{(2)} | \psi^{(2)} \rangle = 0, \quad (5.19)$$

$$\langle \psi^{(0)} | \psi^{(2)} \rangle = -\langle \psi^{(2)} | \psi^{(0)} \rangle = 1. \quad (5.20)$$

Because of the appearance of the operator \mathcal{O}_η , we really should define a second invariant of a manifold M , namely the path integral with an insertion of \mathcal{O}_η , that is the path integral

$$Z_X(M, \mathcal{O}_\eta) = \int \mathcal{D}\phi^i \mathcal{D}\eta^I \mathcal{D}\chi_\mu^I \mathcal{O}_\eta \exp(-S). \quad (5.21)$$

with an insertion of \mathcal{O}_η at some (immaterial) point on M . The evaluation of this partition function is elementary; the insertion of \mathcal{O}_η absorbs the η zero modes, so the whole analysis of

Section 3 collapses to the product of one-loop determinants with no Feynman diagrams at all to correct them. Therefore $Z_X(M, \mathcal{O}_\eta) = 0$ if $b_1(M) > 0$, as there is then no way to absorb the χ zero modes, while if M is a rational homology sphere then, with the normalization (5.18),

$$Z_X(M, \mathcal{O}_\eta) = |H_1(M, \mathbf{Z})|. \quad (5.22)$$

In fact, this formula is valid for any 3-manifold M because of the definition (3.20) of $|H_1(M, \mathbf{Z})|$.

Now let us use our knowledge of $\mathcal{H}_{S^2}(\text{K3})$ in order to determine the behavior of $Z_{\text{K3}}(M)$ under the operation of connected sum. We will first derive this behavior by an argument similar to that of [1] and then look at path integrals. Ignore for a moment the definitions (5.16) and (5.17). Define $\psi^{(0)}$ and $\psi^{(2)}$ as elements of a 2-dimensional space $\mathcal{H}_{S^2}(\text{K3})$, which are produced by the sigma-model path integral taken over a 3-ball B^3 without or with an insertion of \mathcal{O}_η :

$$|\psi^{(0)}\rangle = |B^3\rangle, \quad |\psi^{(2)}\rangle = |B^3, \mathcal{O}_\eta\rangle. \quad (5.23)$$

If we glue two 3-balls along their boundary S^2 , then we get a 3-sphere S^3 . Therefore the scalar products $\langle\psi^{(0)}|\psi^{(0)}\rangle$, $\langle\psi^{(0)}|\psi^{(2)}\rangle$ and $\langle\psi^{(2)}|\psi^{(2)}\rangle$ are equal to the path integral over S^3 with 0,1 or 2 insertions of \mathcal{O}_η . Since

$$\lambda_C(S^3) = 0, \quad |H_1(S^3, \mathbf{Z})| = 1, \quad (5.24)$$

and for any M , the path integral with *two* insertions of \mathcal{O}_η is zero,

$$Z_{\text{K3}}(M, \mathcal{O}_\eta, \mathcal{O}_\eta) = 0, \quad (5.25)$$

we deduce immediately the scalar products (5.19), (5.20). Note that the operator \mathcal{O}_η is antisymmetric, and hence

$$Z_{\text{K3}}(S^3, \mathcal{O}_\eta) = \langle B^3 | B^3, \mathcal{O}_\eta \rangle = -\langle B^3, \mathcal{O}_\eta | B^3 \rangle \quad (5.26)$$

Let $B^3 \subset M$ be a 3-ball inside a manifold M . Since the space $\mathcal{H}_{S^2}(\text{K3})$ is 2-dimensional, the path integral over $M \setminus B^3$ is a linear combination of the states $\psi^{(0)}$ and $\psi^{(2)}$. To derive

the coefficients of this combination, we observe that gluing $M \setminus B^3$ with B^3 restores the manifold M . The scalar products (5.19), (5.20) imply that

$$|M \setminus B^3\rangle = Z_{K3}(M) |\psi^{(2)}\rangle + |H_1(M, \mathbf{Z})| |\psi^{(0)}\rangle. \quad (5.27)$$

In a similar way one can deduce that the path integral over $M \setminus B^3$ with one insertion of \mathcal{O}_η is equal to

$$|M \setminus B^3, \mathcal{O}_\eta\rangle = |H_1(M, \mathbf{Z})| |\psi^{(2)}\rangle. \quad (5.28)$$

A manifold with the opposite orientation has the same value of $|H_1(M, \mathbf{Z})|$ and an opposite value of $Z_X(M)$, so

$$|M^*\rangle = -Z_{K3}(M) |\psi^{(2)}\rangle + |H_1(M, \mathbf{Z})| |\psi^{(0)}\rangle, \quad |M^*, \mathcal{O}_\eta\rangle = |H_1(M, \mathbf{Z})| |\psi^{(2)}\rangle. \quad (5.29)$$

Now we can work out the general formula for the partition function of a connected sum of manifolds. By definition, if we glue two manifolds $M_1 \setminus B^3$ and $M_2 \setminus B^3$ along their common boundary S^2 then we get the *connected sum* of M_1 and M_2 : $M = M_1 \# M_2$. Since, according to eq. (5.1), gluing corresponds to taking the scalar product, we have

$$Z_{K3}(M_1 \# M_2) = \langle (M_1 \setminus B^3)^* | M_2 \setminus B^3 \rangle = |H_1(M_1, \mathbf{Z})| Z_{K3}(M_2) + |H_1(M_2, \mathbf{Z})| Z_{K3}(M_1). \quad (5.30)$$

This relation is in line with eq. (5.4) and with the gluing property of Casson's invariant:

$$\lambda_C(M_1 \# M_2) = |H_1(M_1, \mathbf{Z})| \lambda_C(M_2) + |H_1(M_2, \mathbf{Z})| \lambda_C(M_1). \quad (5.31)$$

We can likewise compute the other invariant $Z_{K3}(M_1 \# M_2, \mathcal{O}_\eta)$ of the connected sum by gluing $M_1 \setminus B^3$ with $M_2 \setminus B^3$, the latter containing the operator \mathcal{O}_η :

$$Z_{K3}(M_1 \# M_2, \mathcal{O}_\eta) = \langle (M_1 \setminus B^3)^* | M_2 \setminus B^3, \mathcal{O}_\eta \rangle = |H_1(M_1, \mathbf{Z})| |H_1(M_2, \mathbf{Z})|. \quad (5.32)$$

This relation is consistent with eq. (5.30) and the gluing property of the order of the first homology

$$|H_1(M_1 \# M_2, \mathbf{Z})| = |H_1(M_1, \mathbf{Z})| |H_1(M_2, \mathbf{Z})|. \quad (5.33)$$

Though deduced for $X = \text{K3}$, these formulas should hold if X is a more general four-dimensional hyper-Kähler manifold such as X_{AH} , for reasons that have been explained. All that happens if X is changed is that the partition function Z is multiplied by a constant. Notice, for instance, that (5.30) is homogeneous in Z and so remains unchanged if Z is rescaled.

Let us now derive the formulas (5.27), (5.28) from the definitions (5.16), (5.17) by a direct calculation of the path integral (3.3) over $M \setminus B^3$. We fix the boundary values of the fields ϕ^i and η^I

$$\phi^i(x)\big|_{x \in \partial(M \setminus B^3)} = \varphi_0^i, \quad \eta^I(x)\big|_{x \in \partial(M \setminus B^3)} = \eta_0^I, \quad (5.34)$$

and decompose the fields $\phi^i(x)$ and $\eta^I(x)$ according to eq. (3.4) and

$$\eta^I(x) = \eta_0^I + \tilde{\eta}^I(x). \quad (5.35)$$

The fields $\varphi_0^i(x)$ and $\tilde{\eta}^I(x)$ satisfy the zero boundary conditions

$$\varphi^i(x)\big|_{x \in \partial(M \setminus B^3)} = 0, \quad \tilde{\eta}^I(x)\big|_{x \in \partial(M \setminus B^3)} = 0, \quad (5.36)$$

which exclude the zero modes. The absence of $\tilde{\eta}^I$ zero modes means that if $b_1(M) = 0$, then the path integral has a non-zero one-loop contribution. Its absolute value is equal to the Reidemeister torsion of $M \setminus B^3$. However, since the zero-modes of the Reidemeister torsion of a manifold with a boundary are not paired up by the Poincaré duality, the sign cannot be chosen canonically. To fix a sign, one has to choose a particular orientation on the space of zero modes. Our assumption, that the pure one-loop contribution to the path integral is equal to $|H_1(M, \mathbf{Z})| |\psi^{(0)}\rangle$, amounts to such a choice.

To assess the contribution of Feynman diagrams, we substitute the decomposition (5.35) into the vertices (3.28), (3.29). There are no zero modes to absorb the fields $\tilde{\eta}^I(x)$, so they can be set equal to zero in the vertices. Thus the contribution to the path integral over $M \setminus B^3$ comes from the same Feynman diagrams that contributed to $Z_{\text{K3}}(M)$, except that the fields η^I in their vertices (3.28), (3.29) should be substituted by the boundary values η_0^I , and the integral over $d^{4n}\varphi_0^i$ should not be taken. As a result, the contribution becomes a function

of the boundary values φ_0^i, η_0^I . One can easily see that this function is equal to $Z_{K3}(M)\psi^{(2)}$ with $\psi^{(2)}$ defined by eq. (5.18), if one uses the second line in the definition (5.18) of \mathcal{O}_η . Note, however, that both definitions (5.18) coincide for $X = K3$. Thus we are ultimately led to eq. (5.27). Note also, that since both the pure one-loop and the Feynman diagram contributions are proportional to the Reidemeister torsion, their relative sign is fixed and does not depend on any arbitrary choice.

To derive eq. (5.28), one has to substitute the decomposition (5.35) into the operator \mathcal{O}_η and set the fields $\tilde{\eta}^I(x)$ equal to zero. The contribution to the path integral is purely one-loop. Since it is also proportional to the Reidemeister torsion, the relative sign of the right hand side in eqs. (5.27) and (5.28) is fixed.

This completes what we will say in genus zero.

5.2 The Hilbert space of a 2-torus and the formula for a Dehn surgery on a knot

We now consider the case that Σ is a two-torus T^2 . The space of harmonic one-forms on T^2 is two-dimensional. We choose two basic one-forms $\omega^{(1)}, \omega^{(2)}$ such that for a pair of basic one-cycles $C_{1,2}$ of the torus,

$$\int_{C_\beta} \omega^{(\alpha)} = \delta_{\alpha\beta}. \quad (5.37)$$

The corresponding operators χ_α^I satisfy the anti-commutation relations (5.6):

$$\{\chi_\alpha^I, \chi_\beta^J\} = \epsilon^{IJ} \epsilon_{\alpha\beta}. \quad (5.38)$$

The representation of the relation (5.38) is completely similar to that of (5.5). We let χ_2^I act by multiplication operators and we set

$$\chi_1^I = \epsilon^{IJ} \frac{\partial}{\partial \chi_2^J}. \quad (5.39)$$

The representation space is formed by the vacuum $|0\rangle_{\chi_2}$ annihilated by χ_1^I :

$$\chi_1^I |0\rangle_{\chi_2} = 0, \quad (5.40)$$

together with the states produced by the operators χ_2^I :

$$\chi_2^{I_1} \cdots \chi_2^{I_l} |0\rangle_{\chi_2}. \quad (5.41)$$

There is just one crucial difference from the case of η zero modes: to represent the quantum states as functions of χ_2 , or functions of χ_1 , requires a choice of a distinguished direction in T^2 , so that the mapping class group $SL(2, \mathbf{Z})$ of T^2 will act quite non-trivially.

The states obtained by quantizing the zero modes now look like

$$|\psi\rangle = \psi_{I_1 \dots I_l, J_1 \dots J_m}(\phi) \chi_2^{I_1} \cdots \chi_2^{I_l} \eta^{J_1} \cdots \eta^{J_m} |0\rangle_{\eta\chi}. \quad (5.42)$$

The states, in other words, are sections of $\wedge^* V \otimes \wedge^* V$. Upon picking one of the complex structures on X , $\wedge^* V$ can be identified with $\Omega^{0,*}(X)$, which is also isomorphic, given the hyper-Kähler structure on X , to $\Omega^{*,0}(X)$. One can thus identify $\wedge^* V \otimes \wedge^* V$ as $\Omega^{*,*}(X)$. The scalar product between the (l, m) and $(2n - l, 2n - m)$ forms is defined as

$$\langle \psi^{(1)} | \psi^{(2)} \rangle = \frac{1}{(2\pi)^{2n}} \int_X \sqrt{g} d^{4n} \varphi_0^i \epsilon^{I_1 \dots I_{2n}} \epsilon^{\bar{J}_1 \dots \bar{J}_{2n}} \psi_{I_1 \dots I_l, \bar{J}_m \dots \bar{J}_1}^{(1)} \psi_{I_{l+1} \dots I_{2n}, \bar{J}_{m+1} \dots \bar{J}_{2n}}^{(2)}. \quad (5.43)$$

Again, one of the Q 's coincides with the $\bar{\partial}$ operator, so if X is compact, the space of physical states is

$$\mathcal{H}_{T^2}(X) = \bigoplus_{l,m=1}^{2n} H^{l,m}(X) \quad (5.44)$$

(cf. eq. (5.8)). If X is non-compact, the continuous spectrum starting at zero energy obstructs a reduction to a description with a finite-dimensional space of physical states. We therefore consider only compact X , such as $X = K3$, to obtain the surgery formulas.

The mapping class group of T^2 is $SL(2, \mathbf{Z})$. A matrix

$$U = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2, \mathbf{Z}), \quad ps - qr = 1 \quad (5.45)$$

transforms a pair of bosonic cycles (C_1, C_2) as

$$U : \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \mapsto \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \quad (5.46)$$

According to eq. (5.37), the pairs of operators (χ_1^I, χ_2^I) are transformed by the same matrix. For instance, χ_1^I maps to $p\chi_1^I + q\chi_2^I$. We have represented χ_2^I by multiplication and χ_1^I by $\epsilon^{IJ}\partial/\partial\chi_2^J$. The state $|0\rangle_{\chi_2}$ which obeys (5.40) is mapped by U to a state

$$|0\rangle_{p,q} = p^n \exp\left(-\frac{q}{p} \epsilon_{IJ} \chi_2^I \chi_2^J\right) |0\rangle_{\chi_2} \quad (5.47)$$

with

$$\left(p\epsilon^{IJ} \frac{\partial}{\partial\chi_2^J} + q\chi_2^I\right) |0\rangle_{p,q} = 0. \quad (5.48)$$

As χ_2^I maps to $r\chi_1^I + s\chi_2^I$, a general state $f(\chi_2)|0\rangle_{\chi_2}$ is mapped to

$$f(r\epsilon^{IJ}(\partial/\partial\chi_2^J) + s\chi_2^I)|0\rangle_{p,q}. \quad (5.49)$$

As a special case of this, an upper triangular matrix with $q = 0$ maps $|0\rangle_{\chi_2} = |0\rangle_{1,0}$ to itself and maps $f(\chi)|0\rangle_{\chi_2}$, with f of degree m , to $(f(\chi) + g(\chi))|0\rangle_{\chi_2}$, where g is of degree less than m . Also, the matrix

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (5.50)$$

exchanges “filled states” with “empty states,” mapping $f(\chi)|0\rangle_{1,0}$ with f homogeneous of degree l to $\tilde{f}(\chi)|0\rangle_{1,0}$ with \tilde{f} homogeneous of degree $2n - l$. To be somewhat more precise, for a subset of indices $\mathcal{A} \in \{1, \dots, 2n\}$, S maps a monomial $\prod_{I \in \mathcal{A}} \chi_2^I$ to a dual monomial $\pm \prod_{I \notin \mathcal{A}} \chi_2^I$.

It is easy to make this explicit for the case that X is of dimension four. The interesting example is of course X a K3 surface. The Hodge diamond of K3 is well-known:

$$\begin{array}{ccccccc} & & h^{0,0} & & & & 1 \\ & & & & h^{0,1} & & 0 & 0 \\ h^{1,0} & & & & & & = & 1 & 20 & 1 \\ & h^{2,0} & h^{1,1} & h^{0,2} & & & & & & \\ & & h^{2,1} & h^{1,2} & & & 0 & 0 & & \\ & & & h^{2,2} & & & & 1 & & \end{array} \quad (5.51)$$

Here $h^{p,q}$ is the dimension of $H^{p,q}(X)$.

It is obvious from the construction of the space \mathcal{H}_{T^2} that the action of $SL(2, \mathbf{Z})$ does not change the number of η 's, which is m , so $\bigoplus_l H^{l,m}$ will furnish an $SL(2, \mathbf{Z})$ representation for each fixed m . From results in the preceding paragraph, one can see that $SL(2, \mathbf{Z})$ acts trivially on $H^{1,1}$, which is twenty-dimensional, but because of the trivial $SL(2, \mathbf{Z})$ action will play only a limited role in what follows. As for $H^{l,m}$ with $l, m = 0, 2$, these groups are all one-dimensional. From the results of the last paragraph, the matrix S can be seen to map $H^{0,m}$ to $H^{2,m}$ and vice-versa. Likewise, an upper triangular matrix leaves $H^{0,m}$ invariant while mapping $H^{2,m}$ to $H^{2,m} \oplus H^{0,m}$. From these facts it follows that $H^{0,m} \oplus H^{2,m}$, for each $m = 0, 2$, transforms as the two-dimensional representation of $SL(2, \mathbf{Z})$.

The knowledge of the structure of the Hilbert space $\mathcal{H}(T^2)$ allows us to present an alternative way of calculating the partition function of a torus bundle over a circle T_U , which was considered in the end of subsection 4.2. From the Hilbert space point of view, the invariant $Z_{K3}(T_U)$ can be calculated as a super-trace of the matrix U represented in \mathcal{H}_{T^2} :

$$Z_{K3}(T_U) = \text{STr}_{\mathcal{H}_{T^2}(K3)} U \quad (5.52)$$

The fermionic parity of a subspace $H^{l,m}(X)$ is $(-1)^{l+m}$, so the states in $\mathcal{H}_{T^2}(K3)$ are bosonic. Since the Hilbert space $\mathcal{H}_{T^2}(K3)$ splits into 20 trivial and 2 fundamental representations of $SL(2, \mathbf{Z})$, we have according to eq. (5.52),

$$Z_{K3}(T_U) = 2(p + s) + 20, \quad (5.53)$$

which is in full agreement with eqs. (4.37) and (5.3). It is attractive that the number $20 = \dim H^{1,1}(X)$ appears here. In particular, when $U = I$, the manifold T_U is isomorphic to $T^2 \times S^1$, and eq. (5.53) tells us that

$$Z_{K3}(T^2 \times S^1) = 24. \quad (5.54)$$

This is in agreement with eqs. (4.43) and (5.3).

Let us choose as follows vectors $\psi^{(l,m)}$, $l, m = 0, 2$ in the spaces $H^{l,m}$. Choose $\psi^{(0,0)}$ and $\psi^{(0,2)}$ as in the previous subsection:

$$\psi^{(0,0)} = 1, \quad \psi_{\bar{I}_1 \bar{I}_2}^{(0,2)} = g_{\bar{I}_1 J_1} g_{\bar{I}_2 J_2} \epsilon^{J_1 K_1} \epsilon^{J_2 K_2} \omega_{K_1 K_2}; \quad (5.55)$$

here the $(2, 0)$ -form ω_{IJ} is defined by eq. (5.18). Define the forms $\psi^{(2,0)}$ and $\psi^{(2,2)}$ as

$$\psi_{I_1 I_2}^{(2,0)} = -\frac{1}{2}\epsilon_{I_1 I_2}\psi^{(0,0)}, \quad \psi_{I_1 I_2, \bar{J}_1 \bar{J}_2}^{(2,2)} = -\frac{1}{2}\epsilon_{I_1 I_2}\psi_{\bar{J}_1 \bar{J}_2}^{(0,2)}. \quad (5.56)$$

The form $\psi^{(2,2)}$ can be expressed equivalently as

$$\psi_{I_1 I_2, \bar{J}_1 \bar{J}_2}^{(2,2)} = \frac{1}{b_\theta(K3)} g_{\bar{J}_1 N_1} g_{\bar{J}_2 N_2} \epsilon^{N_1 K_1} \epsilon^{N_2 K_2} \epsilon^{L_1 L_2} \epsilon^{M_1 M_2} \Omega_{I_1 K_1 L_1 M_1} \Omega_{I_2 K_2 L_2 M_2}. \quad (5.57)$$

The scalar products of the states (5.55) and (5.56) are calculated with the help of eq. (5.43):

$$\begin{aligned} \langle \psi^{(0,0)} | \psi^{(2,2)} \rangle &= \langle \psi^{(2,2)} | \psi^{(0,0)} \rangle = 1, \\ \langle \psi^{(2,0)} | \psi^{(0,2)} \rangle &= \langle \psi^{(0,2)} | \psi^{(2,0)} \rangle = -1. \end{aligned} \quad (5.58)$$

The transformation (5.45) acts on these basis vectors as

$$U : \begin{pmatrix} \psi^{(0,*)} \\ \psi^{(2,*)} \end{pmatrix} \mapsto \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \psi^{(0,*)} \\ \psi^{(2,*)} \end{pmatrix}, \quad (5.59)$$

Now we can use Feynman diagrams in order to calculate the states produced on the torus boundary of a 3-manifold. Let \mathcal{K} be a knot in a 3-manifold M . For the sake of simplicity, we assume that M is a rational homology sphere and that the homology class of \mathcal{K} is trivial. We want to determine a state $|M \setminus \mathcal{K}\rangle$ produced by the path integral over M minus a tubular neighborhood of \mathcal{K} .

First of all, we have to set the boundary conditions on $T^2 = \partial(M \setminus \mathcal{K})$. We make the following choice of basic cycles on T^2 : C_1 is the meridian of the knot complement, that is, the cycle which is contractible through $M \setminus \mathcal{K}$. C_2 is the parallel of the knot complement, that is, the cycle which is contractible through the tubular neighborhood of \mathcal{K} . C_1 and C_2 have a unit intersection number. We impose the following boundary conditions on the fields:

$$\begin{aligned} \phi^i(x) \Big|_{x \in \partial(M \setminus \mathcal{K})} &= \varphi_0^i, & \eta^I(x) \Big|_{x \in \partial(M \setminus \mathcal{K})} &= \eta_0^I, \\ \chi_\mu^I(x) \Big|_{x \in \partial(M \setminus \mathcal{K})} &= \chi^I \omega_\mu^{(2)}(x), & \mu &= 1, 2, \end{aligned} \quad (5.60)$$

Here $\omega^{(2)}$ is one of the forms (5.37) and $\mu = 1, 2$ means that we fix only the components of $\chi_\mu^I(x)$ which are (co-)tangent to T^2 .

We off-set the boundary conditions by splitting the fields according to eqs. (3.4), (5.35) and

$$\chi_\mu^I(x) = \chi^I \omega_\mu^{(2)}(x) + \tilde{\chi}_\mu^I(x). \quad (5.61)$$

Here we use the same notation $\omega_\mu^{(2)}$ for a closed one-form which is an extension of $\omega_\mu^{(2)}$ from T^2 to $M \setminus \mathcal{K}$. The fields $\varphi^i(x), \tilde{\eta}^I(x), \tilde{\chi}_\mu^I(x)$ satisfy boundary conditions

$$\varphi^i(x)|_{x \in \partial(M \setminus \mathcal{K})} = 0, \quad \tilde{\eta}^I(x)|_{x \in \partial(M \setminus \mathcal{K})} = 0, \quad \tilde{\chi}_\mu^I(x)|_{x \in \partial(M \setminus \mathcal{K})} = 0, \quad \mu = 1, 2, \quad (5.62)$$

which exclude the zero modes. Therefore the one-loop contribution to the path integral over $M \setminus \mathcal{K}$ is equal to $|H_1(M, \mathbf{Z})|$ and the pure one-loop contribution to the state $|M \setminus \mathcal{K}\rangle$ is $|H_1(M, \mathbf{Z})| |\psi^{(0,0)}\rangle$.

Apart from the purely one-loop contribution to $|M \setminus \mathcal{K}\rangle$, there are two contributions which come from Feynman diagrams. The first contribution comes from the θ -graph. The fields $\eta^I(x)$ in its vertices are substituted by the boundary values η_0^I coming from eq. (5.35), while the fields $\chi_\mu^I(x)$ are substituted by the fields $\tilde{\chi}_\mu^I(x)$ of eq. (5.61). A contribution of this graph was considered in the previous subsection, it is equal to $Z_{K3}(M) |\psi^{(0,2)}\rangle$.

The second Feynman diagram contribution comes from the same graph as in subsection 4.3, except that the fields $\eta^I(x)$ in both of its vertices are substituted by η_0^I , one field $\chi_\mu^I(x)$ in each vertex is substituted by $\chi^I \omega_\mu^{(2)}(x)$ and the remaining fields $\chi_\mu^I(x)$ are substituted by $\tilde{\chi}_\mu^I(x)$. The contribution of this Feynman diagram can be computed with the help of the same trick that was used in subsection 4.3. Namely, this diagram appears in the calculation of the contribution of the flat connection

$$A_\mu^a = \delta_3^a u \omega_\mu^{(2)}(x) \quad (5.63)$$

(cf. eq. (4.28)) into the Chern-Simons partition function of the knot complement $M \setminus \mathcal{K}$. As explained in [26], this partition function produces the Jones polynomial of the knot \mathcal{K} with the (uK) -dimensional representation of $SU(2)$ assigned to it. The graph that we need represents a part of the one-loop contribution of the connection (5.63) that comes from the fields (4.29) and is proportional to u^2 after the contribution of the zero modes is factored

out. The full one-loop contribution of the fields (4.29) is equal to the inverse Reidemeister torsion of $M \setminus \mathcal{K}$: $[\tau_R(M \setminus \mathcal{K}; e^{2iu})]^{-1}$ (see, e.g. [9] and references therein). The expansion of the Reidemeister torsion at small u is

$$\tau_R(M \setminus \mathcal{K}; e^{2iu}) = \frac{|H_1(M, \mathbf{Z})|}{2iu} \left(1 + \sum_{n=1}^{\infty} C_n(M \setminus \mathcal{K}) u^{2n} \right). \quad (5.64)$$

The prefactor $1/(2iu)$ is due to zero modes, so the quadratic contribution to the inverse Reidemeister torsion is equal to $-C_1(M \setminus \mathcal{K})$. After the same algebra as in subsection 4.3, we conclude that the remaining contribution to the state $|M \setminus \mathcal{K}\rangle$ is

$$-\frac{1}{4}b_\theta(\text{K3}) \left(iu\tau_R(M \setminus \mathcal{K}; e^{iu}) \right)''_{u=0} |\psi^{(2,2)}\rangle \quad (5.65)$$

(the state $|\psi^{(2,2)}\rangle$ is produces in its form (5.57)).

The Reidemeister torsion of the knot complement is related to the Alexander polynomial of the knot:

$$\tau_R(M \setminus \mathcal{K}; t) = \frac{|H_1(M, \mathbf{Z})|}{t^{1/2} - t^{-1/2}} \Delta_A(\mathcal{K}; t). \quad (5.66)$$

Therefore the state (5.65) can be expressed in terms of the second derivative $\Delta_A''(\mathcal{K}) = (\Delta_A(\mathcal{K}; t))''_{t=1}$ as

$$\frac{1}{4}b_\theta(\text{K3})|H_1(M, \mathbf{Z})| \left(\Delta_A''(\mathcal{K}) - \frac{1}{12} \right) |\psi^{(2,2)}\rangle. \quad (5.67)$$

After adding up all three contributions, we find that the state created by a complement of a knot \mathcal{K} inside a rational homology sphere M is

$$\begin{aligned} |M \setminus \mathcal{K}\rangle &= |H_1(M, \mathbf{Z})| |\psi^{(0,0)}\rangle + Z_{\text{K3}}(M) |\psi^{(0,2)}\rangle \\ &\quad + \frac{1}{4}b_\theta(\text{K3})|H_1(M, \mathbf{Z})| \left(\Delta_A''(\mathcal{K}) - \frac{1}{12} \right) |\psi^{(2,2)}\rangle. \end{aligned} \quad (5.68)$$

Let us insert an operator \mathcal{O}_η into the knot complement $M \setminus \mathcal{K}$. The only contribution to the state $|M \setminus \mathcal{K}, \mathcal{O}_\eta\rangle$ is purely one-loop, and both fields $\eta^I(x)$ in \mathcal{O}_η should be substituted by η_0^I . Therefore

$$|M \setminus \mathcal{K}, \mathcal{O}_\eta\rangle = |H_1(M, \mathbf{Z})| |\psi^{(0,2)}\rangle. \quad (5.69)$$

If we reverse the orientation of M , then $Z_{K3}(M)$ changes its sign. Thus

$$|(M \setminus \mathcal{K})^*\rangle = |H_1(M, \mathbf{Z})| |\psi^{(0,0)}\rangle - Z_{K3}(M) |\psi^{(0,2)}\rangle \quad (5.70)$$

$$+ \frac{1}{4} b_\theta(K3) |H_1(M, \mathbf{Z})| \left(\Delta''_A(\mathcal{K}) - \frac{1}{12} \right) |\psi^{(2,2)}\rangle.$$

$$|(M \setminus \mathcal{K})^*, \mathcal{O}_\eta\rangle = |H_1(M, \mathbf{Z})| |\psi^{(0,2)}\rangle. \quad (5.71)$$

Let us use the formulas (5.68)–(5.71) together with the scalar product (5.58) and $SL(2, \mathbf{Z})$ representation (5.59) in order to derive the surgery properties of the partition function $Z_{K3}(M)$. First of all, we apply eq. (5.68) to the complement of an unknot in S^3 . This complement is isomorphic to a solid torus $D^2 \times S^1$, so

$$|D^2 \times S^1\rangle = |\psi^{(0,0)}\rangle - \frac{1}{48} b_\theta(K3) |\psi^{(2,2)}\rangle. \quad (5.72)$$

If we glue two solid tori together in an obvious way, then we obtain $S^2 \times S^1$. Since $(D^2 \times S^1)^*$ is isomorphic to $D^2 \times S^1$, then according to eq. (5.1),

$$Z_{K3}(S^2 \times S^1) = \langle D^2 \times S^1 | D^2 \times S^1 \rangle = -\frac{1}{24} b_\theta(K3). \quad (5.73)$$

This result is consistent with eq. (4.34).

Now let us perform a rational p/q surgery on a homologically trivial knot \mathcal{K} in a rational homology sphere M . We take a solid torus $D^2 \times S^1$, twist its boundary by a product of matrices SU defined by eqs. (5.45) and (5.50), and then glue it to the boundary of $M \setminus \mathcal{K}$, thus constructing a new closed manifold M' . The matrix S is needed to reconcile the definitions of a meridian and a parallel for the boundaries of $M \setminus \mathcal{K}$ and $D^2 \times S^1$. The partition function of the new manifold M' is a matrix element of SU :

$$Z_{K3}(M', \text{fr}') = \langle (M \setminus \mathcal{K})^* | SU | D^2 \times S^1 \rangle \quad (5.74)$$

$$= p Z_{K3}(M) - \frac{1}{4} b_\theta(K3) |H_1(M, \mathbf{Z})| \left[q \left(\Delta''_A(\mathcal{K}) - \frac{1}{12} \right) + \frac{1}{12} r \right].$$

The presence of the framing symbol fr' in the left hand side of eq. (5.74) reflects the fact that if the original manifold M has a canonical framing, then the framing fr' of the new

manifold M' is not necessarily canonical. In fact, the framing correction (4.20) for the U surgery is

$$\Delta \text{fr}' = -12s(p, q) + \frac{p+s}{q} - 3 \text{sign}(pq). \quad (5.75)$$

Here $s(p, q)$ is the Dedekind sum. Therefore in order to test the relation (5.4), in accordance with eq. (4.18), we have to compare eq. (5.74) with the following modification of Walker's surgery formula:

$$\begin{aligned} \lambda_C(M') + \frac{1}{12}|H_1(M', \mathbf{Z})|\Delta \text{fr}' \\ = \text{sign}(p) \left\{ p\lambda_C(M) + |H_1(M, \mathbf{Z})| \left[q \left(\Delta_A''(\mathcal{K}) - \frac{1}{12} \right) + \frac{1}{12}r \right] \right\}. \end{aligned} \quad (5.76)$$

Comparing eqs. (5.4), (5.74) and (5.76), we see that they are compatible except for the sign factor $\text{sign}(p)$ which is missing in eq. (5.74). This means that if we do a U surgery with $p < 0$, then there is an extra -1 appearing in eq. (4.23).

We suggest the following explanation for the “anomalous” sign factor $\text{sign}(p)$. The relation (4.23) between the partition function $Z_{X_{\text{AH}}}(M, \text{fr})$ and the Casson invariant λ_C , as derived in Section 4, depends on the choice of the sign for the vacuum expectation values (3.21) and (3.22). This choice is related to a choice of orientations on the spaces (3.24). Although these spaces have canonical orientations, a surgery on a manifold M with a canonical orientation dictates an orientation for the spaces on the new manifold M' . If $p < 0$ (or if $p = 0$ and $q > 0$), then the surgery-induced orientation differs from the canonical one, and the partition function $Z_{X_{\text{AH}}}(M, \text{fr})$ picks up an extra negative sign. This situation is reminiscent of the framing anomaly of the Chern-Simons gauge theory.

That the sign factor of the origin just described is precisely $\text{sign}(p)$ should be demonstrated directly, but we will instead just test this claim by a surgery calculation of the more elementary invariant $|H_1(M, \mathbf{Z})|$. Applying eq. (5.69) to a solid torus $D^2 \times S^1$, we find that

$$|D^2 \times S^1, \mathcal{O}_\eta\rangle = |\psi^{(0,2)}\rangle. \quad (5.77)$$

Then a combination of eqs. (5.2) and (5.71) leads to the formula

$$Z_{K3}(M', \mathcal{O}_\eta) = \langle (M \setminus \mathcal{K})^* | SU | D^2 \times S^1, \mathcal{O}_\eta \rangle = p |H_1(M, \mathbf{Z})|. \quad (5.78)$$

The surgery formula for $|H_1(M', \mathbf{Z})|$ is

$$|H_1(M', \mathbf{Z})| = \text{sign}(p) p |H_1(M, \mathbf{Z})|, \quad (5.79)$$

Comparing eqs. (5.22), (5.78) and (5.79) we see that the same factor $\text{sign}(p)$ is missing also in eq. (5.78). This happens for the same reason as in eq. (5.74).

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Appendix

After this paper was submitted, M. Kontsevich [27] and M. Kapranov [28] showed that the definition of the weight functions $b_{\Gamma}(X)$ requires less structure on the manifold X than we have assumed so far. Namely, X does not necessarily have to be hyper-Kähler. It is enough for X to be complex and to have a holomorphic symplectic structure. The purpose of this appendix is to explain how this generalization can be understood in the framework of the topological sigma-model.

We will construct a sigma-model whose target is a complex manifold X which reduces in the hyper-Kähler case to the construction given in Section 2. Given a choice of local complex coordinates on X , a map from a three-manifold M to X is described by bosonic fields $\phi^I(x^\mu)$ and $\bar{\phi}^{\bar{I}}(x^\mu)$ describing the map from the 3-manifold M to X in local coordinates. The model will also have two fermionic fields $\chi_\mu^I(x^\mu)$ and $\eta^{\bar{I}}(x^\mu)$ which are respectively a one-form and zero-form which take values in the fibers of the pull-back of the holomorphic and anti-holomorphic tangent bundles of X respectively. (To compare with the construction given in Section 2 for the hyper-Kähler case, the reader should note that on a hyper-Kähler manifold there is a natural isomorphism between the holomorphic and anti-holomorphic tangent bundles of X .)

We will construct a model with a single fermionic symmetry that we will call \overline{Q} . Its action on the fields will be

$$\begin{aligned}\delta\phi^I &= 0, & \delta\overline{\phi}^{\overline{I}} &= \eta^{\overline{I}}, \\ \delta\eta^I &= 0, & \delta\chi_\mu^I &= -\partial_\mu\phi^I\end{aligned}\tag{A.1}$$

We will introduce \overline{Q} -invariant lagrangians L_1 and L_2 that parallel those of Section 2. The construction of L_2 requires some extra structure on X . Let Γ_{JK}^I be a symmetric connection in the holomorphic tangent bundle of X : $\Gamma_{JK}^I = \Gamma_{KJ}^I$. The (1,1)-part of the curvature associated with Γ_{JK}^I represents the so-called Atiyah class of X

$$R_{JK\overline{L}}^I = \frac{\partial\Gamma_{JK}^I}{\partial\overline{\phi}^{\overline{L}}}.\tag{A.2}$$

We also assume that X has a holomorphic symplectic structure ϵ_{IJ} . The (2,0)-form ϵ_{IJ} does not have to be covariantly constant with respect to the connection Γ_{JK}^I . We will only use the fact that ϵ_{IJ} is non-degenerate and closed

$$\frac{\partial\epsilon_{IJ}}{\partial\overline{\phi}^{\overline{K}}} = 0, \quad \frac{\partial\epsilon_{IJ}}{\partial\phi^K} + \frac{\partial\epsilon_{KI}}{\partial\phi^J} + \frac{\partial\epsilon_{JK}}{\partial\phi^I} = 0.\tag{A.3}$$

A \overline{Q} -invariant lagrangian L_2 is a slight modification of the lagrangian (2.15)

$$L_2 = \frac{1}{2} \frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} \left(\epsilon_{IJ} \chi_\mu^I \nabla_\nu \chi_\rho^J - \frac{1}{3} \epsilon_{IJ} R_{KL\overline{M}}^J \chi_\mu^I \chi_\nu^K \chi_\rho^L \eta^{\overline{M}} + \frac{1}{3} (\nabla_L \epsilon_{IK}) (\partial_\mu \phi^I) \chi_\nu^K \chi_\rho^L \right).\tag{A.4}$$

Here ∇_μ is a covariant derivative with respect to the pull-back of the connection Γ_{JK}^I

$$\nabla_\mu \chi_\nu^I = \partial_\mu \chi_\nu^I + (\partial_\mu \phi^J) \Gamma_{JK}^I \chi_\nu^K.\tag{A.5}$$

The BRST-class of L_2 is independent of the choice of connection Γ_{JK}^I . Indeed, if we change a connection Γ_{JK}^I by a tensor A_{JK}^I

$$\Gamma_{JK}^I \rightarrow \Gamma_{JK}^I + A_{JK}^I,\tag{A.6}$$

then L_2 is changed by a BRST-exact term

$$L_2 \rightarrow L_2 + \delta \left(\frac{1}{3} \epsilon_{IJ} A_{KL}^J \chi_\mu^I \chi_\nu^K \chi_\rho^L \right).\tag{A.7}$$

A construction of the \overline{Q} -exact lagrangian L_1 requires a choice of an hermitian metric $g_{I\overline{J}}$ on X . We use the notations

$$\tilde{\Gamma}_{\overline{JK}}^{\overline{I}} = \frac{1}{2} g^{\overline{I}L} \left(\frac{\partial g_{L\overline{J}}}{\partial \overline{\phi}^{\overline{K}}} + \frac{\partial g_{L\overline{K}}}{\partial \overline{\phi}^{\overline{J}}} \right), \quad \tilde{T}_{\overline{JK}}^{\overline{I}} = \frac{1}{2} g^{\overline{I}L} \left(\frac{\partial g_{L\overline{J}}}{\partial \overline{\phi}^{\overline{K}}} - \frac{\partial g_{L\overline{K}}}{\partial \overline{\phi}^{\overline{J}}} \right), \quad (\text{A.8})$$

here $\tilde{\Gamma}_{\overline{JK}}^{\overline{I}}$ is a symmetric connection and $\tilde{T}_{\overline{JK}}^{\overline{I}}$ is a torsion associated with $g_{I\overline{J}}$. Then L_1 can be written as a BRST commutator

$$L_1 = \overline{Q} \left(g_{I\overline{J}} \chi_\mu^I (\partial_\mu \overline{\phi}^{\overline{J}}) \right) = g_{I\overline{J}} \partial_\mu \phi^I \partial_\mu \overline{\phi}^{\overline{J}} + g_{I\overline{J}} \chi_\mu^I \tilde{\nabla}_\mu \eta^{\overline{J}}, \quad (\text{A.9})$$

here $\tilde{\nabla}_\mu$ is a covariant derivative with respect to the connection $\tilde{\Gamma}_{\overline{JK}}^{\overline{I}} + \tilde{T}_{\overline{JK}}^{\overline{I}}$

$$\tilde{\nabla}_\mu \eta^{\overline{I}} = \partial_\mu \eta^{\overline{I}} + (\partial_\mu \overline{\phi}^{\overline{J}}) (\tilde{\Gamma}_{\overline{JK}}^{\overline{I}} + \tilde{T}_{\overline{JK}}^{\overline{I}}) \eta^{\overline{K}}. \quad (\text{A.10})$$

If $g_{I\overline{J}}$ is a Kähler metric, then $\tilde{T}_{\overline{JK}}^{\overline{I}} = 0$ and the perturbative calculations in the sigma-model are similar to those of Section 3. In particular, the relevant interaction vertices contained in the lagrangians (A.4) and (A.9) are

$$V_1 = -\frac{1}{6} \frac{1}{\sqrt{\hbar}} \epsilon^{\mu\nu\rho} \epsilon_{IJ} R_{KL\overline{M}}^J \chi_\mu^I \chi_\nu^K \chi_\rho^L \eta^{\overline{M}} \quad (\text{A.11})$$

$$V_2 = g_{I\overline{J}} \tilde{R}_{\overline{KL}M}^{\overline{J}} \chi_\mu^I \eta^{\overline{L}} (\partial_\mu \overline{\phi}^{\overline{K}}) \varphi^M, \quad (\text{A.12})$$

here $\tilde{R}_{\overline{JKL}}^{\overline{I}}$ is the curvature tensor associated with the connection $\tilde{\Gamma}_{\overline{JK}}^{\overline{I}}$

$$\tilde{R}_{\overline{JKL}}^{\overline{I}} = \frac{\partial \tilde{\Gamma}_{\overline{JK}}^{\overline{I}}}{\partial \phi^L}. \quad (\text{A.13})$$

The formula for the weight function $b_\Gamma(X)$ is derived from the vertex (A.12) similarly to how it was done in subsection 3.4. The holomorphic indices of the curvature tensors $R_{JK\overline{L}}^I$ assigned to the vertices of a closed $2n$ -vertex graph $\Gamma \in \Gamma_{n,3}$, are contracted with the (inverse) symplectic form ϵ^{IJ} . After anti-symmetrizing over the anti-holomorphic indices, we obtain a $\overline{\partial}$ -closed $(0, 2n)$ -form on X . In other words, we get a map

$$\Gamma_{n,3} \rightarrow H_{\overline{\partial}}^{2n}(X). \quad (\text{A.14})$$

In order to get a weight $b_\Gamma(X)$ we wedge-multiply the $(0, 2n)$ -form, which is the image of a graph Γ , by the $(2n, 0)$ -form $\epsilon^{I_1 \dots I_{2n}}$ and integrate the resulting $(2n, 2n)$ -form over X .

The weight function $b_\Gamma(X)$ satisfies the IHX relation, because the $(0, 2n)$ -form which corresponds to the image of an IHX combination of graphs, is $\bar{\partial}$ -exact and thus belongs to the kernel of the map (A.14). This follows from one of the Bianchi identities satisfied by the curvature tensor $R^I_{JK\bar{L}}$ which shows that the Atiyah class is a $\bar{\partial}$ -closed form

$$\frac{\partial}{\partial \bar{\phi}^{\bar{M}}} R^I_{JK\bar{L}} = \frac{\partial}{\partial \bar{\phi}^{\bar{L}}} R^I_{JK\bar{M}}. \quad (\text{A.15})$$

This identity implies that

$$\begin{aligned} \bar{\partial} \left(\nabla_M R^I_{JK\bar{L}} \right) &\equiv \frac{\partial}{\partial \bar{\phi}^{\bar{N}}} \left(\nabla_M R^I_{JK\bar{L}} \right) - (\bar{N} \leftrightarrow \bar{L}) \\ &= \left[\frac{\partial}{\partial \bar{\phi}^{\bar{N}}}, \nabla_M \right] R^I_{JK\bar{L}} + \nabla_M \left(\frac{\partial}{\partial \bar{\phi}^{\bar{N}}} R^I_{JK\bar{L}} \right) - (\bar{N} \leftrightarrow \bar{L}) \\ &= R^I_{PM\bar{N}} R^P_{JK\bar{L}} - R^P_{JM\bar{N}} R^I_{PK\bar{L}} - R^P_{KM\bar{N}} R^I_{JP\bar{L}} - (\bar{N} \leftrightarrow \bar{L}) \quad (\text{A.16}) \\ &= R^I_{PM\bar{N}} R^P_{JK\bar{L}} + R^I_{PK\bar{N}} R^P_{JM\bar{L}} + R^I_{JP\bar{N}} R^P_{KM\bar{L}} - (\bar{N} \leftrightarrow \bar{L}), \end{aligned}$$

that is, the right hand side of eq. (A.16) is $\bar{\partial}$ -exact.

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